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THE STRONG MAXIMUM PRINCIPLE REVISITED

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Abstract. In this paper we first present the classical maximum principle due to E. Hopf, together with an extended commentary and discussion of Hopf's paper. We emphasize the comparison technique invented by Hopf to prove this principle, which has since become a main mathematical tool for the study of second order elliptic partial differential equations and has generated an enormous number of important applications. While Hopf's principle is generally understood to apply to linear equations, it is in fact also crucial in nonlinear theories, such as those under consideration here.

In particular, we shall treat and discuss recent generalizations of the strong maximum principle, and also the compact support principle, for the case of singular quasilinear elliptic differential inequalities, under generally weak assumptions on the quasilinear operators and the nonlinearities involved. Our principal interest is in necessary and sufficient conditions for the validity of both principles; in exposing and simplifying earlier proofs of corresponding results; and in extending the conclusions to wider classes of singular operators than previously considered.

The results have unexpected ramifications for other problems, as will develop from the exposition, e.g.

(i) two point boundary value problems for singular quasilinear ordinary differential equations (Sections 3, 4);

(ii) the exterior Dirichlet boundary value problem (Section 5);

(iii) the existence of dead cores and compact support solutions, i.e. dead cores at infinity (Section 7);

(iv) Euler–Lagrange inequalities on a Riemannian manifold (Section 9);

(v) comparison and uniqueness theorems for solutions of singular quasilinear differential inequalities (Section 10).

The case of p-regular elliptic inequalities is briefly considered in Section 11.

Contents

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1. Introduction

The strong maximum principle of Eberhard Hopf is a classical and bedrock result of the theory of second order elliptic partial differential equations. It goes back to the maximum principle for harmonic functions, already known to Gauss in 1839 on the basis of the main value theorem. On the other hand, it carries forward to maximum principles for singular quasilinear elliptic differential inequalities, a theory initiated particularly by Vázquez and Diaz in the 1980's, but with earlier intimations in the work of Benilan, Brezis and Crandall.

Our purpose here is to provide a clear explanation of this type of result, from its beginnings, to show its relation with and differences from the classical theory of Hopf, and to develop the ramifications of these ideas in rather unexpected byways. In particular, there are intimate connections with a number of fundamental questions of elliptic partial differential equations, more specifically in the noteworthy directions:

(i) two point boundary value problems for singular quasilinear ordinary differential equations (Sections 3, 4);

(ii) the exterior Dirichlet boundary value problem (Section 5);

(iii) the existence of dead cores and compact support solutions, i.e. dead cores at infinity (Section 7);

(iv) Euler–Lagrange inequalities on a Riemannian manifold (Section 9);

(v) comparison and uniqueness theorems for solutions of singular quasilinear differential inequalities (Section 10).

These areas and their relevant connections will be developed throughout the course of the article, see especially Sections 3, 4, 5, 7, 9 and 10. We shall particularly emphasize and maintain the nonlinear nature of the operators involved, in contrast to the naive view sometimes expressed that Hopf's original result applies principally to linear operators.

After an initial discussion of the maximum principle of Eberhard Hopf, Section 2, we shall turn our attention in the following sections especially to the strong maximum principle and the compact support principle for quasilinear differential inequalities. To introduce these questions in the most natural way, it is convenient first to describe a canonical type of inequality to which the discussion applies, and to clarify the structure of these model inequalities by means of special examples.

Thus we consider in the first instance the strong maximum principle and the compact support principle for quasilinear elliptic differential inequalities, under generally weak assumptions on the quasilinear operators in question, in the canonical divergence structure

$$\operatorname{div}\{A(|Du|)Du\} - f(u) \le 0, \qquad u \ge 0, \tag{1.1}$$

and

$$\operatorname{div}\{A(|Du|)Du\} - f(u) \ge 0, \qquad u \ge 0, \tag{1.2}$$

in a domain (connected open set) Ω in $\mathbb{R}^n$, $n \ge 2$. Here Du denotes the vector gradient of the given function $u = u(x)$, $x \in \mathbb{R}^n$. We assume throughout the paper, unless otherwise stated explicitly, the following conditions on the operator $A = A(\rho)$ and the nonlinearity $f = f(u)$,

(A1) $A \in C(0,\infty)$,
(A2) $\rho \mapsto \rho A(\rho)$ is strictly increasing in $(0,\infty)$ and $\rho A(\rho) \to 0$ as $\rho \to 0$;
(F1) $f \in C[0,\infty)$,

(F2) $f(0) = 0$ and f is non–decreasing on some interval $(0, \delta)$, $\delta > 0$.

Condition (A2) is a minimal requirement for ellipticity of (1.1)–(1.2). Furthermore, it allows singular and degenerate behavior of the operator A at $\rho = 0$, that is at critical points of u. We emphasize that no assumptions of differentiability are made on either A or f when dealing with the canonical models (1.1) and (1.2).

The operator $\mathrm{div}\{A(|Du|)Du\}$ will be called the *A–Laplace operator*, so as to place it in the context of well–known elliptic theory.

By a *classical solution* (or a classical distribution solution) of (1.1) or (1.2) in Ω we mean a non–negative function $u \in C^1(\Omega)$ which satisfies (1.1) or (1.2) in the distribution sense.

With the notation $\Phi(\rho) = \rho A(\rho)$ when $\rho > 0$, and $\Phi(0) = 0$, we introduce the function

$$H(\rho) = \rho\Phi(\rho) - \int_0^\rho \Phi(s)ds, \qquad \rho \geq 0. \tag{1.3}$$

This function is easily seen to be strictly increasing, as follows from the inequality

$$\rho_1\Phi(\rho_1) - \rho_0\Phi(\rho_0) > (\rho_1 - \rho_0)\Phi(\rho_1) > \int_{\rho_0}^{\rho_1} \Phi(s)ds$$

when $\rho_1 > \rho_0 \geq 0$. Alternatively, monotonicity follows from the representation

$$H(\rho) = \int_0^{\Phi(\rho)} \Phi^{-1}(\omega)d\omega, \qquad \rho \geq 0, \tag{1.4}$$

this being a consequence of the Stieltjes formula $H(\rho) = \int_0^\rho s\, d\Phi(s)$.

For the Laplace operator, that is when (1.1) takes the classical form

$$\Delta u - f(u) \leq 0, \qquad u \geq 0,$$

we have $A(\rho) \equiv 1$ and $H(\rho) = \frac{1}{2}\rho^2$. Similarly, for the degenerate p–Laplace operator, here denoted by Δ_p, $p > 1$, we have $A(\rho) = \rho^{p-2}$ and $H(\rho) = (p-1)\rho^p/p$, while for the mean curvature operator, one has $A(\rho) = 1/\sqrt{1+\rho^2}$ and $H(\rho) = 1 - 1/\sqrt{1+\rho^2}$. In the last example, note the anomalous behavior $\Phi(\infty) = H(\infty) = 1$, a possibility which occasionally requires extra care in the statement and treatment of results.

It is also worth observing that (1.1), when equality holds, is precisely the Euler–Lagrange equation for the variational integral

$$I[u] = \int_\Omega \{\mathcal{G}(|Du|) + F(u)\}dx, \qquad F(u) = \int_0^u f(s)ds, \tag{1.5}$$

where $\mathcal{G}$ and A are related by $A(\rho) = \mathcal{G}'(\rho)/\rho$, $\rho > 0$. In this case $H(\rho) = \rho\mathcal{G}'(\rho) - \mathcal{G}(\rho)$, the pre–Legendre transform of $\mathcal{G}$. Further comments and other examples of operators satisfying (A1), (A2) are given in [30].

By the strong maximum principle for (1.1) we mean the statement that *if u is a classical solution of* (1.1) *with $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u \equiv 0$ in Ω.*

We can now state the main results of [27], which are proved in Section 6 using a very much simplified method based on the results of Sections 3, 4 and 5.

Theorem 1.1. (Strong maximum principle). *In order for the strong maximum principle to hold for* (1.1) *it is necessary and sufficient either that $f(s) \equiv 0$ for $s \in [0, \mu)$, $\mu > 0$, or that $f(s) > 0$ for $s \in (0, \delta)$ and*

$$\int_0^\delta \frac{ds}{H^{-1}(F(s))} = \infty. \tag{1.6}$$

As is well known, the strong maximum principle is extremely useful when studying the qualitative behavior of solutions of differential equations and inequalities. The choice of the base level zero for the statement of the principle is of course a matter only of convenience, as is whether we deal with minimum or maximum values at the base point x_0.

The background and literature for Theorem 1.1 is fairly complicated and deserves a number of comments:

The necessity of (1.6) for the case of the Laplace operator is due to Benilan, Brezis and Crandall [4], while for the p–Laplacian it is due to Vázquez [41]. In these cases we observe that (1.6) reduces respectively to

$$\int_0^\delta \frac{ds}{\sqrt{F(s)}} = \infty \quad \text{and} \quad \int_0^\delta \frac{ds}{[F(s)]^{1/p}} = \infty.$$

For general operators satisfying (A1), (A2), necessity is due to Diaz ([11], Theorem 1.4), see also ([30], Corollary 1).

Sufficiency for the case of the Laplace operator and also for the p–Laplacian is again due to Vázquez [41], see also [11] and [38]. For general operators satisfying (A1), (A2), sufficiency was proved in Theorem 1 of [30] under an additional technical assumption, and in Theorem 1 of [27] without the technical assumption.

The case when $f \equiv 0$ was studied by Cellina [5] for non–negative minimizers of the integral $\int_\Omega \mathcal{G}(|Du|)dx$. An alternative abstract approach to the strong maximum principle appears in [6].

The regular case. If $A(\rho)$ is continuous on $[0,\infty)$, $\lim_{\rho\to 0} A(\rho) = \alpha > 0$, and $f(u) \leq \text{Const.}\, u$, $(u \geq 0)$, then clearly $\Phi(\rho) \approx \alpha\rho$ and $H(\rho) \approx \alpha\rho^2/2$ for small ρ, while also $F(u) \leq \text{Const.}\, u^2$; thus obviously the strong maximum principle is valid. In fact, far stronger results are known in this direction [36]:

Let u and v be classical distribution solutions of the differential inequalities

$$\begin{aligned} &\operatorname{div}\hat{\boldsymbol{A}}(x,u,Du) - \hat{B}(x,u,Du) \leq 0\\ &\operatorname{div}\hat{\boldsymbol{A}}(x,v,Dv) - \hat{B}(x,v,Dv) \geq 0, \end{aligned}$$

in Ω, where the vector function $\hat{\boldsymbol{A}}(x,z,\boldsymbol{\xi})$ and the scalar $\hat{B}(x,z,\boldsymbol{\xi})$ are continuously differentiable in the variables z, $\boldsymbol{\xi}$, and the matrix $[\partial\hat{\boldsymbol{A}}/\partial\boldsymbol{\xi}]$ is positive definite for all values of its variables.

If $u \geq v$ in Ω, then either $u \equiv v$ or $u > v$ in Ω.

We shall not pursue this direction further, since our interest is essentially in functions $\hat{\boldsymbol{A}}$ and $\hat{B}$ which are singular or degenerate, respectively when $Du = \mathbf{0}$ and when $u = 0$.

A rigorous treatment of the full sufficiency result of Theorem 1.1, avoiding use of the technical assumption (2.5) of [30], is not entirely obvious, involving as it does the solution of differential inequalities whose structure includes driving and amplifying terms which reinforce each other. The proof here uses only standard calculus, and the elementary Leray–Schauder theorem (see [18], Theorem 11.6), but requires neither monotone operator theory (as [41], [11]–[14]), nor Orlicz–Sobolev space theory (as [23]), nor viscosity solution theory (as [21]), nor probabilistic methods. The proofs have further applications as well, for example to dead core theory, see Section 7 and uniqueness for the Dirichlet problem, see Section 9.

In the next result we consider the situation when the integral in (1.6) is convergent. Here the appropriate hypotheses are that u satisfies the converse inequality (1.2) and also "vanishes" at ∞, rather than at some finite point $x_0 \in \Omega$.

More precisely, by the compact support principle for (1.2) we mean the statement that *if u is a classical solution of* (1.2) *in an exterior domain Ω, with $u(x) \to 0$ as $|x| \to \infty$, then u has compact support in Ω.*

Theorem 1.2. (Compact support principle). *Let $f(u) > 0$ for $0 < u < \delta$. Then in order for the compact support principle to hold for* (1.2) *in an exterior domain Ω, it is* *necessary and sufficient that*

$$\int_0^\delta \frac{ds}{H^{-1}(F(s))} < \infty. \tag{1.7}$$

As in the case of the strong maximum principle it is worth commenting on the background and literature for Theorem 1.2.

Necessity was first shown in Corollary 2 of [30] under the additional technical assumption (2.5) of [30], and in [27], with a proof which is in totality not at all easy. The proof given here is simpler and at the same time provides an existence theorem for radial solutions of exterior Dirichlet problems, see Theorem 5.1.

The sufficiency of (1.7) is Theorem 2 of [30], but see also [31] and the remarks following the statement of Theorem 2 in [30]. For radially symmetric solutions of (1.2) sufficiency was proved in [17] under the weaker assumption that $F(s) > 0$ for $s \in (0, \delta)$, see Proposition 1.3.1 of [17].

If Theorem 1.2 were an exact analogue of Theorem 1.1, the conclusion of the compact support principle would be that $u \equiv 0$ in Ω, but this would be incorrect since (1.2) admits non–trivial compact support solutions under assumption (1.7), see [17] and Theorem 7.5 below.

The existence of compact support solutions for quasilinear equations was studied extensively in the 80's, as well as other properties of the set where the solution u vanishes, for example the case of *dead cores*. In chemical models, when u represents the density of a reactant, the vanishing of a solution then delineates a region where no reactant is present (see [1], [12]). A short discussion of dead cores for (1.1), with equality sign, is given in Section 7, see Theorems 7.2 and 7.3.

The results described above can be extended to a wider class of differential inequalities by replacing div $\{A(|Du|)Du\}$ by the more general operator $D_i\{a_{ij}(x,u)A(|Du|)D_j u\}$ and $f(u)$ by $B(x,u,Du)$, where $[a_{ij}(x,u)]$ is a positive definite symmetric matrix on $\Omega \times \mathbb{R}^+_0$ and where B satisfies a (typical) condition of the form

$$-\text{Const.}\, \Phi(|\boldsymbol{\xi}|) + g(u) \le B(x,u,\boldsymbol{\xi}) \le \text{ Const.}\, \Phi(|\boldsymbol{\xi}|) + f(u) \tag{1.8}$$

for $x \in \Omega$, $u \ge 0$ and all $\boldsymbol{\xi} \in \mathbb{R}^n$ with $|\boldsymbol{\xi}|$ sufficiently small, and with f and g satisfying (F1) and (F2); see Theorem 8.1 and 8.5, and their corollaries, these being the second main goal of the paper; see also Section 9.

An important prototype is the equation

$$\Delta_p u - |Du|^q - f(u) = 0, \qquad p > 1,\ q > 0. \tag{1.9}$$

Since $\Phi(\rho) = \rho^{p-1}$ for this case, condition (1.8) applies with $f = g$ and requires $q \ge p-1$; that is, the strong maximum principle holds for (1.9) when $q \ge p-1$ and either $f \equiv 0$ in $[0,\mu]$, $\mu > 0$, or f obeys (1.6) – see Corollary 8.3. On the other hand, when $q \in (0, p-1)$ the strong maximum principle can fail, even when $f \equiv 0$, e.g. the C^1 function $u(x) = C|x|^k$ satisfies

$$\Delta_p u - |Du|^q = 0, \tag{1.10}$$

where

$$k = \frac{p-q}{s}, \qquad \frac{1}{C} = k\left[\frac{(p-1)n-(n-1)q}{s}\right]^{1/s}, \qquad s = p-1-q > 0$$

(for $p = 2$, this example is due to Barles, Diaz and Diaz [3]). It is of further interest in connection with this example that the compact support principle can fail even if (1.8) is satisfied, namely when $q > p-1$! Indeed, the function $u(x) = L|x|^{-l}$ satisfies (1.10) in

$\Omega_R = \mathbb{R}^n \setminus \overline{B_R}$, with $l = (p-q)/t > 0$, provided that

$$q > \frac{n(p-1)}{n-1}, \qquad L = \frac{1}{l}\left[\frac{(n-1)q-(p-1)n}{t}\right]^{1/t}, \qquad t = q-p+1.$$

For the special case when $A \equiv 1$ and $f(u) = u^q$, $q > 0$, the strong maximum principle holds for non–negative C^1 distribution solutions of $\Delta u - u^q \le 0$ if and only if $q \ge 1$, while the compact support principle holds for non–negative C^1 distribution solutions of $\Delta u - u^q \ge 0$ if and only if $0 < q < 1$. Actually by the main results of [17], or by Section 7 below, there exist C^2 non–negative radially symmetric compact support solutions of $\Delta u - u^q = 0$ when $0 < q < 1$.

Note that when $q = 0$ our analysis cannot be applied. Let $c \in \mathbb{R}$. The strong maximum principle holds for non–negative C^1 distribution solutions of $\Delta u - c \le 0$ only if $c \le 0$. Indeed the equation $\Delta u - 2n = 0$ in any domain Ω of $\mathbb{R}^n$ containing the origin admits the non–trivial solution $u(x) = |x|^2$, but $u(0) = 0$. We also note that the equation $\Delta u - c = 0$, with $c \ne 0$, admits no compact support solutions no matter what of the sign of c, as follows from the Hopf boundary point lemma.

The same remarks apply to the p–Laplacian analogue $\Delta_p u - u^q = 0$, $p > 1$ and $q > 0$, for which the compact support principle holds for non–negative C^1 distribution solutions if and only if $0 < q < p-1$, while the strong maximum principle holds if and only if $q \ge p-1$.

As we shall note in Section 2, dedicated to the original work of E. Hopf (see also [37]), the Maximum Principle implies the Comparison Principle, Theorem 2.4. On the other hand, for singular equations, even if they are smooth, the situation is more delicate. Consider for example

$$\Delta_4 u + |Du|^2 = 0, \qquad n = 2, \tag{1.11}$$

which, when expanded to the form $\mathcal{F}(Du, D^2u) = 0$ is smooth (even analytic), elliptic when $Du \ne \mathbf{0}$, and degenerate,[1] that is, $\partial\mathcal{F}/\partial(D^2u) = 0$ when $Du = \mathbf{0}$. The Strong Maximum Principle continues to hold (see Theorem 8.1), while on the other hand (1.11) admits two unequal solutions $u \equiv 0$ and $u(x) = \frac{1}{8}(R^2 - |x|^2)$ in B_R, both with the same boundary values.

The paper is structured as follows. In Section 2 we present the classical Hopf Maximum Principle together with some comments of independent interest. Section 3 is devoted to some preliminary lemmas, and Section 4 to existence and uniqueness for related two point boundary value problems for quasilinear ordinary differential equations.

Section 5 deals with the existence and uniqueness of classical radial solutions of the exterior Dirichlet problem for (1.1), or (1.2), with equality sign, namely for the case of equations. The results are important in the proof of the compact support principle, but are also of independent interest.

In Section 6 we prove the main Theorems 1.1 and 1.2 for the canonical models (1.1) and (1.2).

In Section 7 the existence of dead cores for (1.1), with equality sign, is proved, and also the existence of compact support solutions of (1.1) in exterior domains.

In Sections 8.1 and 8.2 we consider the case of fully quasilinear inequalities

$$D_i\{a_{ij}(x,u)A(|Du|)D_ju\} - B(x,u,Du) \le 0 \ \ (\ge 0), \qquad u \ge 0 \tag{1.12}$$

[1] In particular, in this case

$$\mathcal{F}(Du, D^2u) = |Du|^2\Delta u + 2\sum_{i,j=1}^{2} D_iuD_juD^2_{ij}u + |Du|^2.$$

(where the obvious summation convention is used). Section 9 extends these considerations to the quasilinear inequality

$$D_i\{a_{ij}(x,u)A(|Du|_g)D_ju\} - B(x,u,Du) \le 0, \tag{1.13}$$

where $|Du|_g = \sqrt{g^{ij}(x,u)D_iuD_ju}$ is a gradient norm of Riemannian type, an important case which includes variational problems on Riemannian manifolds; in this regard we emphasize here particularly Theorem 9.3.

Section 10 contains a series of general comparison principles for singular elliptic inequalities of divergence type. These results, which extend well–known theorems of Gilbarg and Trudinger, are important not only in proving our main conclusions for the strong maximum principle, but naturally are useful well beyond this application. In particular, they imply various uniqueness results for the Dirichlet problem, see e.g. Theorems 10.8 and 10.10, which appear to be new in the generality given.

Section 11 contains a brief discussion of the strong maximum principle for p–regular inequalities, alternative to the previous considerations.

Finally, in Section 12 we treat several special cases where the main proof of Proposition 4.1 reduces to a simpler form. As a byproduct of this discussion we obtain a rational comparison function for some special inequalities, alternative to the classical exponential function of E. Hopf.

2. The Hopf Maximum Principle

Before giving the main results already stated, we present the classical principle due to E. Hopf in [20], together with an extended commentary and discussion of Hopf's original paper by J. Serrin [37].

The maximum principle for harmonic and subharmonic functions was known to Gauss on the basis of the mean value theorem (1839); an extension to elliptic inequalities however remained open until the twentieth century. Bernstein (1904), Picard (1905), Lichtenstein (1912, 1924) then obtained various results by difficult means, as well as use of regularity conditions for the coefficients of the highest order terms. It was Hopf's genius to see that a "gǎnzlich elementare Begrǘnden" could be given. The comparison technique he invented for this purpose is essentially so transparent that it has generated an enormous number of important applications in many further directions.

Here is Hopf's theorem in its main form:

Let $u = u(x)$, $x = (x_1, \dots, x_n)$, be a C^2 function which satisfies the differential inequality

$$Lu \equiv \sum_{i,j} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial u}{\partial x_i} \ge 0$$

in a domain Ω, where the (symmetric) matrix $a_{ij} = a_{ij}(x)$ is locally uniformly positive definite in Ω and the coefficients a_{ij}, $b_i = b_i(x)$ are locally bounded.

If u takes a maximum value M in Ω, then $u \equiv M$ in Ω.

Hopf's proof (Section I of [20]), now a classic of the subject, is reproduced in the monographs [26] and [18], and in many other texts as well, particularly the second volume of [7]. *The hypothesis that u is of class C^2 is essential for the theorem, though not always strictly noted in presentations of the result.* For maximum principles when u is not of class C^2, and even possibly only measurable, see e.g. Littman [22]; for the case of C^1 distribution solutions, see the later results of the present paper, as discussed in the introduction.

Hopf next observes (Section II of [20]) that one can allow the coefficients to depend on the solution u itself, provided that when they are evaluated along a solution the resulting functions $a_{ij}(x)$, $b_i(x)$ satisfy the conditions of the main theorem. This allows him to deal explicitly with nonlinear as well as linear equations.

In the same section he then notices two important corollaries (Sǎtze 2, 3) dealing with the differential inequality $Lu + cu \ge 0$. *First, for the case $c = c(x) \le 0$ and a positive* *maximum, and second, when there is an extremum $M = 0$ with c being bounded but not necessarily non–positive.* The latter possibility is not mentioned in [18]. Moreover, Courant and Hilbert in their formulation of Satz 2 in [7] do not include the crucial restriction to a positive maximum.

Because Hopf's formulation of these results is somewhat obscure, the main conclusions are worth restating here, which we do in terms of the operator L.

Theorem 2.1. *Let u be a C^2 function satisfying the differential inequality*

$$Lu + cu \geq 0 \quad (\leq 0) \tag{2.1}$$

in a domain Ω, where the coefficients of L satisfy the previous conditions, and $c = c(x)$ is a non-positive function on Ω. If u takes a positive maximum (negative minimum) value M in Ω, then $u \equiv M$.

Theorem 2.2. *Let the hypotheses of Theorem* 2.1 *hold, except that one now assumes only that the function c is locally bounded. If u takes on a vanishing maximum (minimum) value $M = 0$ in Ω, then $u \equiv 0$.*

The real depth of Hopf's nonlinear analysis shows up only in Section III of [20], though the presentation is seriously obscured by the restriction to exact equations, as well as to the case where one of the solutions in question is assumed to vanish identically ("engere Voraussetzungen" according to Hopf). Accordingly we shall again restate the results, in slightly greater generality and in more usual notation.

Theorem 2.3. (Touching Lemma). *Let u, v be $C^2(\Omega)$ solutions of the nonlinear differential inequalities*

$$\mathcal{F}(x, u, Du, D^2u) \geq 0, \qquad \mathcal{F}(x, v, Dv, D^2v) \leq 0,$$

where $\mathcal{F}$ is of class C^1 in the variables u, Du, D^2u (notation obvious). Suppose also that the matrix

$$Q_{ij} \equiv \frac{\partial \mathcal{F}}{\partial (D^2_{ij}u)}(x, u, Du, \theta D^2u + (1-\theta)D^2v)$$

is positive definite in Ω for all $\theta \in [0, 1]$.

If $u \leq v$ in Ω and $u = v$ at some point x_0 in Ω, then $u \equiv v$ in Ω.

The terms u, Du in Q can be replaced by v, Dv.

Proof. Essentially following Hopf's proof of Satz $3'$ of [20], we write

$$\begin{aligned} 0 \geq \mathcal{F}(x, v, Dv, D^2v) - \mathcal{F}(x, u, Du, D^2u) &= \mathcal{F}(x, u, Du, D^2v) - \mathcal{F}(x, u, Du, D^2u) \\ &+ \mathcal{F}(x, u, Dv, D^2v) - \mathcal{F}(x, u, Du, D^2v) + \mathcal{F}(x, v, Dv, D^2v) - \mathcal{F}(x, u, Dv, D^2v) \\ = \sum a_{ij} D^2_{ij}(v-u) + \sum b_i D_i(v-u) &+ c(v-u) = L(v-u) + c(v-u), \end{aligned}$$

where, for some values θ, θ_1, $\theta_2 \in [0, 1]$ we have

$$\begin{aligned} a_{ij} &= \frac{\partial \mathcal{F}}{\partial (D^2_{ij}u)}(x, u, Du, \theta D^2v + (1-\theta)D^2u) = Q_{ij} \\ b_i &= \frac{\partial \mathcal{F}}{\partial D_i u}(x, v, \theta_1 Dv + (1-\theta_1)Du, D^2u) \\ c &= \frac{\partial \mathcal{F}}{\partial u}(x, \theta_2 v + (1-\theta_2)u, Dv, D^2v). \end{aligned}$$

Clearly a_{ij}, b_i, c are locally bounded, and equally by continuity the coefficient matrix a_{ij} is locally uniformly positive definite on Ω. Since by assumption $v - u \geq 0$ and $(v-u)(x_0) = 0$, it now follows from Theorem 2.2 that $v \equiv u$ in Ω.

To obtain the final conclusion of the theorem, one proceeds in the same way, though starting from the alternative decomposition

$$\begin{aligned}0 \geq \mathcal{F}(x,v,Dv,D^2v) - \mathcal{F}(x,u,Du,D^2u) &= \mathcal{F}(x,v,Dv,D^2v) - \mathcal{F}(x,v,Dv,D^2u) \\ + \mathcal{F}(x,v,Dv,D^2u) - \mathcal{F}(x,v,Du,D^2u) &+ \mathcal{F}(x,v,Du,D^2u) - \mathcal{F}(x,u,Du,D^2u).\end{aligned}$$

□

The next result (essentially Satz 2′ of [20] in a more general context and formulation) is stated as a *comparison result*, rather than a maximum principle, this being the underlying content of Hopf's result.

Theorem 2.4. (Comparison Lemma). *Let u, v be $C^2(\Omega) \cap C(\overline{\Omega})$ solutions of the nonlinear differential inequalities given in Theorem 2.3. Suppose that the matrix $Q = Q_{ij}$ is positive definite in Ω and that*

$$\Psi = \frac{\partial \mathcal{F}}{\partial u}(x,w,Dv,D^2v) \leq 0$$

for all functions $w \geq v$ (or simply for all functions w on Ω).

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

The terms u, Du in Q can be replaced by v, Dv if at the same time the terms Dv, D^2v in Ψ are replaced by Du, D^2u and the condition $w \geq v$ is replaced by $w \leq u$.

Proof. Suppose for contradiction that the conclusion $v - u \geq 0$ in Ω fails.

Then there will be a subdomain Ω' of Ω in which $v-u \leq 0$ but is not identically constant, and in which also $v - u$ takes on a negative minimum M. As in the proof of Theorem 2.3 one obtains

$$L(v-u) + c(v-u) \leq 0,$$

while by hypothesis $c \leq 0$ in Ω'. Hence by Theorem 2.1 we get $v - u \equiv M$ in Ω', a contradiction.

The final conclusion is obtained from the alternative decomposition in the proof of Theorem 2.3. □

Using other decompositions, one can obtain various related results, e.g. Theorem 31 of Chapter 2 of [26].

A direct consequence of Theorem 2.4 is a uniqueness theorem for the Dirichlet problem for the nonlinear equation $\mathcal{F}(x,u,Du,D^2u) = 0$, a fact mentioned by Hopf in the final paragraph of [20], though not explicitly formulated by him. Since the result is important, and a precise formulation is in fact not immediate from Hopf's analysis, it is worth stating a definite result here.

Theorem 2.5. *Let u and v be C^2 solutions of the nonlinear equation*

$$\mathcal{F}(x,u,Du,D^2u) = 0$$

in a domain Ω, with $u = v$ on $\partial\Omega$. Suppose Q is positive definite in Ω for all $\theta \in [0,1]$, and $\Psi \leq 0$ in Ω for all functions w. Then $u \equiv v$.

This is an immediate corollary of Theorem 2.4, the main result being used to establish that $u \leq v$ and the final part of the theorem to get $v \leq u$. Here it is crucial that $\Psi \leq 0$ *for all functions w.*

It is surprising that the matrix Q in the hypothesis of Theorem 2.5 is, insofar as its second and third arguments are concerned, to be evaluated solely on the functions u and Du, *without any symmetric reference to v and Dv.*

Indeed specializing Theorem 2.5 to quasilinear equations, we find that for the equation

$$A_{ij}(x,Du)D^2_{ij}u - B(x,u,Du) = 0$$

a sufficient condition for uniqueness is that the matrix $Q_{ij} = A_{ij}(x,Du)$ *needs to be positive definite (i.e. the equation needs to be elliptic) only when evaluated for either one (!) of the solutions u or v*, provided that $B(x,u,\boldsymbol{\xi})$ is a non–decreasing function of u for arbitrary arguments x, $\boldsymbol{\xi}$. This last result (essentially due to Hopf, though not explicitly mentioned or stated by him) seems to have appeared first in [18], first edition, Chapter 8.

The result applies at once to the quasilinear operator

$$\mathcal{F} = (1 + |Du|^2)\Delta u - \sum \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}$$

(mean curvature), since clearly

$$Q_{ij} = (1 + |Du|^2)I_{ij} - D_i u D_j u$$

is positive definite for all values of its arguments. Here of course there is no need to use the full strength of Theorem 2.5. On the other hand, if we consider the Dirichlet problem

$$(1 + |Du|^2)\Delta u - 2\sum \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$$

in Ω, with $u = 0$ on $\partial\Omega$, then the matrix Q is *not* positive definite for arbitrary arguments $D^2 u$. Nevertheless $Q = I$ for the function $u \equiv 0$, whence it follows that this function is the *unique* solution of the stated Dirichlet problem.

A second and more subtle example is the elementary Monge–Ampère equation in $\mathbb{R}^2$

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = g(x, y).$$

Here one checks that

$$Q_{ij}\xi_i\xi_j = \frac{\partial^2 u}{\partial y^2}\xi_1^2 - 2\frac{\partial^2 u}{\partial x \partial y}\xi_1\xi_2 + \frac{\partial^2 u}{\partial x^2}\xi_2^2.$$

The discriminant of Q is then

$$\det Q = \det \mathbb{H}u = \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2,$$

which is precisely $g = g(x, y)$ when evaluated at a solution u.

Suppose in particular that $g > 0$. It is easy to see then, that *any solution u is either everywhere strictly convex or everywhere strictly concave.*

From this, one can check without difficulty that *if u and v are two convex solutions then Q is positive definite for the arguments $D^2_{ij}(\theta u + (1 - \theta)v)$.*

Hence the Dirichlet problem for the elementary Monge–Ampère equation above has at most one convex solution. On the other hand, if u and v are concave solutions, then $-u$ and $-v$ are convex solutions and so, similarly, the Dirichlet problem can have at most one concave solution; altogether then *the problem can have at most two solutions.* This result is a special case of a theorem of Rellich [32]; see [7], page 324.

Other related maximum and comparison principles are discussed in the Notes to Chapter 2 of [26], and in Chapter 10 of [18], to which the reader is strongly referred; see also the references cited on page 314 of [42]. A viscosity based maximum principle for singular fully nonlinear equations is given in [2].

Hopf's proof technique, as noted above, leads to other results of fundamental interest, particularly the celebrated Boundary Point Lemma and a Harnack principle for elliptic equations having two independent variables; for this last result, see the paper [34] of J. Serrin, reproduced in both [26] and [18]. A nonlinear version of the Harnack principle in two variables has also been given recently in [28].

3. Some preliminary lemmas

Here we turn to the study of the strong maximum principle and of the compact support principle for divergence structure quasilinear elliptic operators and for nonlinear terms $f(u)$. In general, the results described cannot be obtained from the nonlinear theorems of the previous section, since the operators and equations in question for the most part have specialized properties which are lost when they are written in the expanded form $\mathcal{F}(x, u, Du, D^2u) = 0$ as required there.

We shall assume from here on, and throughout the paper unless otherwise mentioned explicitly, that A and f satisfy $(A1)$, $(A2)$, $(F1)$, $(F2)$. Moreover, without loss of generality (since we deal with non–negative solutions) one may suppose that

$$f(u) = 0 \quad \text{ for } \; u \le 0.$$

For convenience in what follows it is useful to extend the definition of the principal operator Φ to all values real values of ρ by setting $\Phi(\rho) = -\Phi(-\rho)$ when $\rho < 0$, unless otherwise explicitly specified.

Following and refining [27], we require several preliminary lemmas.

Lemma 3.1. *(i) For any constant $\sigma \in [0, 1]$ there holds*

$$F(\sigma u) \le \sigma F(u), \qquad u \in [0, \delta].$$

(ii) Let $w = w(t)$ be of class $C^1(0,T)$, and write $' = d/dt$. If $\Phi \circ w'$ is of class $C^1(0,T)$ then $H \circ w'$ is of class $C^1(0,T)$, and in this case

$$[H(w'(t))]' = w'(t)[\Phi(w'(t))]' \qquad \text{in } (0,T). \tag{3.1}$$

On the other hand, if $H \circ w'$ is of class $C^1(0,T)$ and $w' > 0$, then $\Phi \circ w'$ is of class $C^1(0,T)$ and (3.1) *continues to be satisfied.*

To obtain (i), observe that $\sigma f(\sigma u) \le \sigma f(u)$ for $u \in [0, \delta]$, since f is non–decreasing. Integrating this relation from 0 to u yields the result.

The first statement of (ii) is an immediate consequence of (1.4). The second part is also a consequence of (1.4) together with a small lemma:

Let I be any interval of $\mathbb{R}$ and let

$$B(t) = \int^{a(t)} b(s)ds, \qquad t \in I,$$

where $B \in C^1(I)$; $a, b \in C(I)$; and $b > 0$. Then $a \in C^1(I)$ and $a' = B'/(b \circ a)$.

This is easily demonstrated by using difference coefficients and the integral mean value theorem to get $\Delta B/\Delta t = b(a+\theta\Delta a)\Delta a/\Delta t$, $0 \le \theta \le 1$. The lemma then follows by dividing by $b(a + \theta\Delta a)$ and letting $\Delta t \to 0$.

Lemma 3.2. *Suppose $f(u) > 0$ for $u > 0$ and (in case $H(\infty) < \infty$) that $F(\delta) < H(\infty)$. If $\tau \ge 1$ and* (1.6) *holds, then also*

$$\int_0^{\delta/\tau} \frac{ds}{H^{-1}(\tau F(s))} = \infty.$$

Similarly, if $0 < \sigma \le 1$ and (1.7) *is satisfied, then*

$$\int_0^{\delta} \frac{ds}{H^{-1}(\sigma F(s))} < \infty.$$

Proof. For small $\varepsilon > 0$, we have by Lemma 3.1 (i), with $\sigma = 1/\tau$,

$$\int_{\varepsilon/\tau}^{\delta/\tau} \frac{ds}{H^{-1}(\tau F(s))} \ge \int_{\varepsilon/\tau}^{\delta/\tau} \frac{ds}{H^{-1}(F(\tau s))} = \frac{1}{\tau}\int_{\varepsilon}^{\delta} \frac{dt}{H^{-1}(F(t))}.$$

Letting $\varepsilon \to 0$ and applying (1.6) gives the first result.

Again by Lemma 3.1 (i),

$$\int_{\varepsilon}^{\delta} \frac{ds}{H^{-1}(\sigma F(s))} \le \int_{\varepsilon}^{\delta} \frac{ds}{H^{-1}(F(\sigma s))} = \frac{1}{\sigma}\int_{\varepsilon\sigma}^{\delta\sigma} \frac{dt}{H^{-1}(F(t))}$$

and the second part now follows by letting $\varepsilon \to 0$ and applying (1.7). $\square$

Lemma 3.3. *Let $T > 0$ and assume*

$$q \in C(0,T), \qquad q > 0 \quad \text{in } (0,T). \tag{3.2}$$

Then every classical distribution solution $w = w(t)$ of the problem ($' = d/dt$)

$$\begin{cases} [\operatorname{sign} w(t)] \cdot [q(t)\Phi(w'(t))]' \ge 0 & \text{in } (0,T), \\ w(0) = 0, \quad w(T) = m > 0 \end{cases} \tag{3.3}$$

is such that

$$w \ge 0, \quad w' \ge 0 \quad \text{in } (0,T). \tag{3.4}$$

Even more there exists $t_0 \in [0,T)$ with the property that

$$w \equiv 0 \quad \text{in } [0,t_0]; \qquad w > 0, \quad w' > 0 \quad \text{in } (t_0,T). \tag{3.5}$$

Proof. We first claim that $w \ge 0$ in $[0,T]$. If the conclusion fails, there would be t_0 and t_1, with $0 \le t_0 < t_1 < T$ such that $w(t_0) = w(t_1) = 0$ and $w < 0$ in (t_0,t_1). Then, multiplying (3.3) by w and integrating on $[t_0,t_1]$ yields by integration by parts (or simply by the distribution meaning of solutions with the test function $w(t)$ on $[t_0,t_1]$)

$$\int_{t_0}^{t_1} q(t)\Phi(w'(t))w'(t)dt \le 0,$$

where the integrand is non–negative by (3.2) and the fact that $\rho\Phi(\rho) > 0$ for $\rho \neq 0$. That is, necessarily $w' \equiv 0$ on $[t_0,t_1]$. Hence $w \equiv 0$ on $[t_0,t_1]$, since $w(t_0) = w(t_1) = 0$. This contradiction proves the claim.

Define the set $J = \{t \in (0,T) \ : \ w'(t) > 0\}$. Then, obviously, $J \neq \emptyset$, since $w(0) = 0$ and $w(T) > 0$, while also J is open in $(0,T)$ since $w \in C^1(0,T)$. Let $t_0 = \inf J$, so $t_0 \in [0,T)$ and $w \equiv 0$ in $[0,t_0]$, since we already know that $w \ge 0$ in $[0,T]$. Now, for any fixed $t \in (t_0,T)$ there obviously exists $t_1 \in (t_0,t)$ such that $w'(t_1) > 0$. By integration of (3.3) on $[t_1,t]$, recalling that $w \ge 0$ on $(0,T)$, we get

$$q(t)\Phi(w'(t)) \ge q(t_1)\Phi(w'(t_1)) > 0$$

by (3.2) and $(A2)$, so that $w' > 0$ on $(t_0,T]$. In turn, by integration, $w > 0$ in (t_0,T), proving (3.5). □

Remark. If in Lemma 3.3 the hypothesis (3.2) is strengthened to

$$q \in C(0,T), \qquad q > 0 \quad \text{in } (0,T), \qquad q \ \text{ non–increasing},$$

then w' is non–decreasing on $[0,T]$ and

$$0 \le w'(0) \le \frac{m}{T}. \tag{3.6}$$

Indeed from (3.3) and (3.4) it follows that $q(t)\Phi(w'(t))$ is non–decreasing, and then since $q(t)$ is non–increasing also $\Phi(w'(t))$ is non–decreasing. But Φ is increasing, so w' is non–decreasing. In turn, w is convex on $[0,T]$ and then (3.6) follows at once since $w(T) = m$.

Lemma 3.4. *Assume*

$$q \in C[0,T], \qquad q > 0 \quad \text{in } (0,T). \tag{3.7}$$

Then along every classical distribution solution w of the problem

$$\begin{cases} [q(t)\Phi(w'(t))]' - q(t)f(w(t)) \le 0 & \text{in } (0,T), \\ w(0) = 0; \qquad 0 \le w \le \delta, \quad w' \ge 0 & \text{in } (0,T), \end{cases} \tag{3.8}$$

there holds

$$\Phi(w'(t)) \le \frac{f(w(t))}{q(t)} \int_0^t q(s)\,ds + \frac{q(0)}{q(t)}\Phi(w'(0+)), \tag{3.9}$$

where $w'(0+)$ is defined as $\limsup_{t\to 0+} w'(t)$.

In particular, if $w'(0) = 0$ then (3.9) *reduces to*

$$\Phi(w'(t)) \le \frac{f(w(t))}{q(t)} \int_0^t q(s)\, ds. \tag{3.10}$$

Proof. Integrating (3.8) on $[\tau, t]$, with $0 < \tau < t < T$, yields

$$q(t)\Phi(w'(t)) - q(\tau)\Phi(w'(\tau)) \le \int_0^t q(s) f(w(s)) ds, \tag{3.11}$$

and (3.9) follows at once by (F2), i.e., $f(w(s)) \le f(w(t))$ since $0 \le w(s) \le w(t) < \delta$, together with the lim sup as $\tau \to 0$. □

Lemma 3.5. *Assume* (3.7) *and*

$$q \in C^1(0,T), \qquad \left(-\frac{q'(s)}{q(s)^2}\right)^+ \int_0^s q(\tau) d\tau \quad \textit{bounded on } (0,t) \textit{ for all } t \in (0,T). \tag{3.12}$$

Then along every classical distribution solution $w \in C^1(0,T)$ of the problem (3.8) *for which $w'(0) = 0$ and the condition*

$$\Phi(w') \textit{ is continuously differentiable} \tag{3.13}$$

is satisfied, [2] *we have*

$$H(w'(t)) \le B(t) F(w(t)), \qquad t \in (0,T), \tag{3.14}$$

where

$$B(t) = 1 + \sup_{s\in(0,t)} \left(-\frac{q'(s)}{q(s)^2} \int_0^s q(\tau) d\tau\right)^+. \tag{3.15}$$

Note that if $q' \ge 0$, then (3.14) *becomes $H(w'(t)) \le F(w(t))$.*

Proof. Denote by E the energy function associated to w in $(0,T)$, namely

$$E(t) = H(w'(t)) - F(w(t)).$$

Since $\Phi(w') \in C^1(0,T)$ by assumption, so also $H(w') \in C^1(0,T)$ by Lemma 3.1 (ii). Then by (3.1) and (3.8) one finds (since distribution derivatives of C^1 functions can be treated as ordinary derivatives)

$$E'(t) = w'\{[\Phi(w'(t))]' - f(w(t))\} \le -\frac{q'(t)}{q(t)} \Phi(w'(t)) w'(t), \qquad t \in (0,T), \tag{3.16}$$

since by assumption $w' \ge 0$, $q > 0$ in $(0,T)$. Integrating (3.16) on $(0,t)$, with $0 < t < T$, yields

$$\begin{aligned} H(w'(t)) &\le F(w(t)) - \int_0^t \frac{q'(s)}{q(s)} \Phi(w'(s)) w'(s) ds \qquad \text{(since } w'(0) = 0\text{)}, \\ &\le F(w(t)) + \int_0^t \left(-\frac{q'(s)}{q(s)^2} \int_0^s q(\tau)\, d\tau\right)^+ f(w(s)) w'(s)\, ds \le B(t) F(w(t)) \end{aligned}$$

by (3.10) and (3.15). □

Proposition 3.6. *Assume* (3.7) *and* (3.12). *Let w be a classical distribution solution of the problem*

$$\begin{cases} [q(t)\Phi(w'(t))]' - q(t) f(w(t)) \le 0 \quad \textit{in } (0,T), \\ w(0) = 0, \quad w(T) = m > 0, \qquad w' \ge 0, \end{cases} \tag{3.17}$$

[2]For the main application of this lemma in Section 4 this condition holds without any difficulty; see Proposition 4.4.

for which (3.13) *is satisfied. Suppose that* $f(u) > 0$ *for* $u > 0$*. If* $w'(0) = 0$ *then*

$$\int_0^\delta \frac{ds}{H^{-1}(F(s))} < \infty. \tag{3.18}$$

Proof. From the second line of (3.17) it is evident that there exists $t_0 \in [0, T)$ such that $w(t) = 0$ for $0 \le t \le t_0$ while $w > 0$ in (t_0, T). If $t_0 = 0$, then $w'(0) = 0$ by hypothesis, while if $t_0 > 0$ then in turn $w(t_0) = w'(t_0) = 0$ since $w \in C^1(0, T)$.

Let $t_2 \in (t_0, T)$. Clearly there exists $t_1 \in (t_0, t_2)$ such that $m_1 = w(t_1) > 0$ satisfies

$$m_1 < \delta/B, \qquad F(Bm_1) < H(\infty),$$

where $B = B(t_2) \ge 1$ is given in Lemma 3.5. From this lemma applied to the interval (t_0, t_1), we thus obtain (see (3.14))

$$H(w'(t)) \le B(t)F(w(t)) \le BF(w(t)) \quad \text{in } (t_0, t_1)$$

since $B(t)$ is obviously non–decreasing. In turn by Lemma 3.1 (i), with $\sigma = 1/B$,

$$H(w'(t)) \le F(Bw(t)) \quad \text{in } (t_0, t_1),$$

that is $w > 0$, $w'(t) \le H^{-1}(F(Bw(t)))$ on (t_0, t_1). Using the fact that $f(u) > 0$ for $u > 0$ (and so also $F(u) > 0$ for $u > 0$), integration now yields

$$\int_0^{Bm_1} \frac{du}{H^{-1}(F(u))} = B\int_0^{m_1} \frac{dw}{H^{-1}(F(Bw))} = B\int_{t_0}^{t_1} \frac{w'(t)dt}{H^{-1}(F(Bw(t)))} \le B(t_1 - t_0) < \infty,$$

as required. □

4. A singular two-point boundary value problem

In this section we shall obtain existence and uniqueness theorems for the differential problems

$$\begin{cases} [q(t)\Phi(w'(t))]' - q(t)f(w(t)) = 0 \quad \text{in } (0, T), \\ w(0) = 0, \quad w(T) = m > 0. \end{cases} \tag{4.1}$$

and

$$\begin{cases} [q(t)\Phi(w'(t))]' - a(t)q(t)f(w(t)) = h(t) \quad \text{in } (0, T), \\ w(0) = 0, \quad w(T) = m > 0. \end{cases} \tag{4.2}$$

The following two main existence theorems, Propositions 4.1 and 4.3, will be crucial in supplying radial comparison functions for the proofs in later sections. Importantly in these propositions, we are able to use a weakened version of condition (F2), namely

(F3) $f(0) = 0$ and f is non–negative on some interval $[0, \overline{\delta})$, with $\overline{\delta}$ possibly infinite.

Accordingly *it will be assumed in both Propositions* 4.1 *and* 4.3 *that* $m \in (0, \overline{\delta})$.

Finally, we shall suppose of the function q in (4.1) and (4.2) that it is of class $C[0, T]$ with $q > 0$ in $[0, T]$. Put

$$q_0 = \min_{[0,T]} q(t) > 0, \qquad q_1 = \max_{[0,T]} q(t) > 0.$$

Of course, in addition to (F3), conditions (A1), (A2), (F1) will be maintained throughout the section.

Proposition 4.1. (*i*). *Let* $\Phi(\infty) = \infty$*. Then problem* (4.1) *admits a classical distribution solution with the properties*

$$w \in C^1[0, T], \quad \Phi(w') \in C^1[0, T]; \quad w' \ge 0. \tag{4.3}$$

Moreover, for any such solution of (4.1) *we have* $w'(T) > 0$ *and*

$$\|w'\|_\infty \le \Phi^{-1}\left(\frac{q_1}{q_0}[T\bar f(m) + \Phi(m/T)]\right), \tag{4.4}$$

where $\bar f(m) = \max_{u\in[0,m]} f(u)$. *In particular,* $w' \le 1$ *if* m *is sufficiently small.*

(*ii*) *Suppose* $\Phi(\infty) = \omega < \infty$. *Let* $m \in (0, \bar\delta)$ *be such that*

$$\frac{q_1}{q_0}[T\bar f(m) + \Phi(m/T)] < \omega. \tag{4.5}$$

Then the conclusion of part (i) continues to hold.

Proof. For the purpose of this proof only, we shall redefine the operator Φ for $\rho < 0$ by setting $\Phi(\rho) = \rho$ when $\rho < 0$; this can be done without loss of generality since the ultimate solution w satisfies $w' \ge 0$.

Case (*i*). Let

$$\mu_1 = q_1[T\bar f(m) + \Phi(m/T)] \tag{4.6}$$

and

$$I = [0, \mu_1].$$

It is convenient also to redefine f so that $f(u) = f(m)$ for all $u \ge m$. This will not affect the conclusion of the proposition, since clearly any ultimate solution with $w' \ge 0$ satisfies $0 \le w \le m$. We recall also the earlier agreement that $f(u) = 0$ for $u \le 0$.

With these preliminaries settled, we can proceed to the main proof. We shall make use of the Leray–Schauder fixed point theorem, an idea suggested in this context by Montenegro.

Denote by X the Banach space $X = C[0,T]$, endowed with the usual norm $\|\cdot\|_\infty$, and let $\mathcal{T}$ be the mapping from X to X defined by [3]

$$\mathcal{T}[w](t) = m - \int_t^T \Phi^{-1}\left(\frac{1}{q(s)}\left[\mu - \int_s^T q(\tau)f(w(\tau))d\tau\right]\right) ds, \qquad t \in [0,T], \tag{4.7}$$

where $\mu = \mu(w) \in I$ is chosen so that

$$\mathcal{T}[w](0) = 0. \tag{4.8}$$

We shall show that such a choice of μ is uniquely possible.

Indeed for any fixed $w \in X$ and for any $\mu \in I$ we have

$$-\frac{\bar f(m)}{q_0}\int_0^T q(t)\,dt \le \frac{1}{q(s)}\left[\mu - \int_s^T q(\tau)f(w(\tau))d\tau\right] \le \frac{\mu_1}{q_0}. \tag{4.9}$$

It follows now that $\mathcal{T}[w]$ is well defined for each fixed μ in I.

Moreover for $\mu = 0$ we see that, for all $w \in X$,

$$\mathcal{T}[w](0) \ge m.$$

On the other hand, for $\mu = \mu_1$ we find, for all w in X,

$$\begin{aligned}\mathcal{T}[w](0) &= m - \int_0^T \Phi^{-1}\left(\frac{q_1}{q(s)}\Phi(m/T) + \frac{1}{q(s)}\left[q_1 T\bar f(m) - \int_s^T q(\tau)f(w(\tau))d\tau\right]\right) ds \\ &\le m - \int_0^T \Phi^{-1}(\Phi(m/T))ds = 0,\end{aligned}$$

[3]The simpler mapping

$$\mathcal{T}[w](t) = \int_0^t \Phi^{-1}\left(\frac{1}{q(s)}\left[\kappa + \int_0^s q(\tau)f(w(\tau))d\tau\right]\right) ds$$

with $\kappa = \kappa(w)$ chosen so that $\mathcal{T}[w](T) = m$, is in fact less convenient in carrying out the proof.

where we have used the condition (4.6), the definition of q_1, and the fact that $0 \le f(u) \le \bar{f}(m)$. Since the integral on the right side of (4.7) is a strictly increasing function of μ for fixed w, it is now obvious that there exists a unique $\mu \in I$ such that (4.8) holds.

Define the homotopy $\mathcal{H} : X \times [0,1] \to X$ by

$$\mathcal{H}[w,\sigma](t) = \sigma m - \int_t^T \Phi^{-1}\left(\frac{1}{q(s)}\left[\mu_\sigma - \sigma\int_s^T q(\tau)f(w(\tau))d\tau\right]\right)ds, \tag{4.10}$$

where $\mu_\sigma = \mu(w,\sigma) \in I$ is a number chosen such that

$$\mathcal{H}[w,\sigma](0) = 0.$$

Clearly, as above, such a value μ_σ exists and is unique, and the mapping $\mathcal{H}[w,\sigma]$ is accordingly well defined.

By construction, any fixed point $w_\sigma = \mathcal{H}[w_\sigma,\sigma]$ is of class $C^1[0,T]$, has the property that $\Phi(w') \in C^1[0,T]$, and is a classical distribution solution of the problem

$$\begin{cases} [q(t)\Phi(w_\sigma'(t))]' - \sigma q(t) f(w_\sigma(t)) = 0 \quad \text{in } [0,T], \\ w_\sigma(0) = 0, \quad w_\sigma(T) = \sigma m. \end{cases} \tag{4.11}$$

Moreover, by Lemma 3.3, a fixed point $w = \mathcal{H}[w,1]$ satisfies $w,\ w' \ge 0$, and so is a solution of problem (4.1) satisfying the conditioins (4.3), with $w' \ge 0$.

It remains to show that such a fixed point $w = w_1$ exists. We shall use Browder's version of the Leray–Schauder theorem for this purpose (see Theorem 11.6 of [18]).

To begin with, obviously $\mu_\sigma = 0$ when $\sigma = 0$, and so $\mathcal{H}[w,0](t) \equiv 0$ for all w in X, that is $\mathcal{H}[w,0]$ maps X into the single point $w_0 = 0$ in X. (This is the first hypothesis required in the application of the Leray–Schauder theorem at the end of the proof.)

We show next that $\mathcal{H}$ is compact and continuous from $X \times [0,1]$ into X. Let $(w_k,\sigma_k)_k$ be a bounded sequence in $X \times [0,1]$. Clearly $\mu_{\sigma_k} \in I$; therefore again using the fact that $0 \le f(u) \le \bar{f}(m)$ for all $u \ge 0$, together with (4.9), it is clear that

$$\|\mathcal{H}'[w_k,\sigma_k]\|_\infty \le C',$$

where (recalling that $\Phi^{-1}(\rho) = \rho$ when $\rho < 0$)

$$C' = \max\left\{\frac{\bar{f}(m)}{q_0}\int_0^T q(t)dt,\ \Phi^{-1}\left(\mu_1/q_0\right)\right\}. \tag{4.12}$$

It is now an immediate consequence of the Ascoli–Arzelà theorem that $\mathcal{H}$ maps bounded sequences into relatively compact sequences in X.

We claim finally that $\mathcal{H}$ is continuous on $X \times [0,1]$. Indeed, let $w_j \to w$, $\sigma_j \to \sigma$, $(w_j,\sigma_j) \in X \times [0,1]$. Then in (4.10) clearly $\sigma_j f(w_j) \to \sigma f(w)$, since the modified function f is continuous[4] on $\mathbb{R}$. It must then be shown that $\mu(w_j,\sigma_j) \to \mu(w,\sigma)$. To this end, suppose for contradiction that this fails. Then, for some subsequence, still called (w_j,σ_j), we should have

$$\mu(w_j,\sigma_j) \to \tilde{\mu} \ne \mu = \mu(w,\sigma).$$

In this case, from (4.8) one gets by subtraction

$$\begin{aligned} \int_0^T \Bigg\{ \Phi^{-1}&\left(\frac{1}{q(s)}\left[\tilde{\mu} - \sigma\int_s^T q(\tau)f(w(\tau))d\tau\right]\right) \\ &- \Phi^{-1}\left(\frac{1}{q(s)}\left[\mu - \sigma\int_s^T q(\tau)f(w(\tau))d\tau\right]\right)\Bigg\} ds = 0 \end{aligned} \tag{4.13}$$

But Φ^{-1} is a monotone increasing function of its argument, so clearly the integrand in (4.13) is either everywhere positive or everywhere negative, giving the required contradiction.

[4]It is here that the condition $f(0) = 0$ in (F3) is crucial. In fact the proposition fails otherwise, as shown by the example $f(u) \equiv 1$, $q \equiv 1$, and $A(\rho) \equiv 1$. In this case every non–negative solution of (4.1) must have the form $w(t) = at + \frac{1}{2}t^2$, $a \ge 0$, which gives the extraneous condition for solvability $m = w(T) = aT + \frac{1}{2}T^2 \ge \frac{1}{2}T^2$.

To apply the Leray–Schauder theorem it is now enough to show that there is a constant $M > 0$ such that

$$\|w\|_\infty \le M \quad \text{for all } (w,\sigma) \in X \times [0,1], \text{ with } \quad \mathcal{H}[w,\sigma] = w. \tag{4.14}$$

Let (w, σ) be a pair of type (4.14). But, as observed above, since $w' \ge 0$, clearly $\|w\|_\infty = w(T) = \sigma m \le m$. Thus we can take $M = m$ in (4.14).

The Leray–Schauder theorem therefore can be applied and the mapping $\mathcal{T}[w] = \mathcal{H}[w, 1]$ has a fixed point $w \in X$, which is the required solution of (4.1). That (4.3) holds for this solution was noted earlier in the proof.

The last part of the theorem is a direct consequence of (4.7) evaluated at a fixed point w, together with the right hand inequality of (4.9) and the fact that $\mu \in I$.

Case (ii). The argument is exactly the same as before, with the single exception that in (4.9) the right hand side μ_1/q_0 is now less than ω by virtue of (4.5). Thus, $\mathcal{T}$ is well–defined in X, and the rest of the proof is unchanged. □

In view of (4.3) we note that, for the given solution w, all derivatives with respect to t in (4.1) can equally well be understood as ordinary derivatives, no recourse to distribution solutions in fact being needed.

The following lemma is important for the next proposition.

Lemma 4.2. *Let condition* (F3) *hold, and assume* $\Phi(\infty) = \infty$. *Suppose also that* $q \in C[0, 1]$ *and that* q *is positive and non–increasing on* $[0, 1]$.

(i) *Let* w *be any solution of* (4.1) *with* $m \in (0, \bar\delta)$ *and* $T = 1$. *Then*

$$w'(1) \le \Phi^{-1}\left(\frac{q(0)}{q(1)}[\bar f(m) + \Phi(m)]\right). \tag{4.15}$$

(ii) *Let* w *be any solution of* (4.1) *with* $m \in (0, \bar\delta)$ *and* $T = 1$, *but now with the initial condition* $w(0) = 0$ *replaced by* w, $w' > 0$ *on* $[0, 1]$. *Then* (4.15) *continues to hold.*

Proof. Case (i) follows from the second part of Proposition 4.1 (i), and the identifications $T = 1$, $q_0 = q(1)$, $q_1 = q(0)$.

The proof of case (ii) lies deeper, relying on an idea in [17].

Let $v = v(t)$ be a solution of (4.1) with $m \in (0, \bar\delta)$ and $T = 1$, given by Proposition 4.1 (i), which exists since $\Phi(\infty) = \infty$ in the present case. Also $q_0 = q(1)$, $q_1 = q(0)$ so that (4.4) implies

$$v'(1) \le \Phi^{-1}\left(\frac{q(0)}{q(1)}[\bar f(m) + \Phi(m)]\right), \tag{4.16}$$

because $T = 1$. We shall show that

$$w'(1) \le v'(1) \tag{4.17}$$

To see this, observe first by Lemma 3.3 that $v \equiv 0$ in $[0, t_0]$; v, $v' > 0$ in $(t_0, 1]$ for some $t_0 \in [0, 1)$. By assumption the given solution w is also such that w, $w' > 0$ in $[0, 1]$. Hence we can introduce the C^1 functions

$$t : [0, m] \to [t_0, 1], \qquad s : [w_0, m] \to [0, 1],$$

respectively inverse to v and w on the sets where v and w are positive; here $w_0 = w(0) > 0$.

Clearly

$$s(w_0) = 0, \quad t(w_0) > t_0; \qquad s(m) = t(m) = 1,$$

and

$$s'(m) = 1/w'(1), \qquad t'(m) = 1/v'(1).$$

If for contradiction (4.17) fails, then $w'(1) > v'(1)$ and $s'(m) < t'(m)$. In this case, we claim that there would be an interval (u_1, m), with $u_1 \in (w_0, m)$, such that

$$s(u) > t(u) > 0, \quad 0 < s'(u) < t'(u) \quad \text{for } u \in (u_1, m); \qquad s'(u_1) = t'(u_1). \tag{4.18}$$

Indeed, since $s(w_0) < t(w_0)$, the condition $s'(u) < t'(u)$, which holds at $u = m$, cannot persist for all smaller values of u in the open interval (w_0, m). Thus there must be a *first point* $u_1 \in (w_0, m)$ where $s'(u_1) = t'(u_1)$, and in turn the claim (4.18) follows at once.

Now by integration of (4.1) along the solution $v(t)$ from $t(u_1)$ to 1, we derive

$$\int_{u_1}^{m} q(t(u))f(u)t'(u)du = \int_{t(u_1)}^{1} q(t)f(v(t))dt = q(1)\Phi(v'(1)) - q(t(u_1))\Phi(v'(t(u_1))),$$

with a similar relation for the solution w. By subtraction

$$\int_{u_1}^{m} [q(t(u))t'(u) - q(s(u))s'(u)]f(u)du = q(1)[\Phi(v'(1)) - \Phi(w'(1))] \\ - [q(t(u_1)) - q(s(u_1))]\Phi(v'(t(u_1))),$$

since $w'(s(u_1)) = v'(t(u_1))$ by (4.18). The left hand side is non–negative by virtue of (F3), condition (4.18), and the fact that q is positive and non–increasing; while the right hand side is negative since $v'(1) < w'(1)$ by the contradiction assumption and again the fact that q is positive and non–increasing. This absurdity shows (4.17), and application of (4.16) then completes the proof. □

Proposition 4.3. *Let q satisfy the conditions given in the paragraph before Proposition* 4.1*, and assume additionally that q is non–increasing.*

(i) Suppose $\Phi(\infty) = \infty$ and let $T \geq 1$, $m \in (0, \overline{\delta})$. Then problem (4.1) *admits a classical distribution solution with $w \in C^1[0, T]$ and $w \geq 0$. Moreover*

$$\|w'\|_\infty \leq \Phi^{-1}\left(\frac{p_1}{p_0}[\bar{f}(m) + \Phi(m)]\right), \tag{4.19}$$

where $p_0 = q(T)$, $p_1 = q(T - 1)$.

(ii) Suppose $\Phi(\infty) = \omega < \infty$ and $T \geq 1$. Let $m \in (0, \overline{\delta})$ be such that

$$\frac{p_1}{p_0}[\bar{f}(m) + \Phi(m)] < \omega. \tag{4.20}$$

Then the conclusion of part (i) continues to hold.

Proof. (i) Consider the auxiliary problem

$$\begin{cases} [q(t)\Phi(v'(t))]' - q(t)f(v(t)) = 0 & \text{in } (T-1, T), \\ v(T-1) = 0, \quad v(T) = m, \end{cases} \tag{4.21}$$

where $m \in (0, \overline{\delta})$. We assert that (4.21) has a $C^1[T - 1, T]$ solution with $v' \geq 0$ and

$$\|v'\|_\infty \leq \Phi^{-1}\left(\frac{p_1}{p_0}[\bar{f}(m) + \Phi(m)]\right). \tag{4.22}$$

The existence in fact follows at once from Proposition 4.1 (i). To prove (4.22), it is enough to translate to the present case the estimate (4.4) in Proposition 4.1 (i). But for this we have obviously

$$q_0 = \min_{[T-1,T]} q(t) = q(T) = p_0, \qquad q_1 = \max_{[T-1,T]} q(t) = q(T - 1) = p_1,$$

since q is non–increasing. Moreover, in (4.5) the length of the interval $[T - 1, T]$ is of course just 1. Hence (4.4) becomes exactly (4.22), as required.

We now apply the comparison Lemma 4.2 to the solution w of Proposition 4.1 (i) and the solution v of (4.21) just determined. Their common interval of definition is just $[T - 1, T]$, an interval of precisely length 1. Clearly $w(T) = v(T) = m$. Moreover either $w(T - 1) = 0$ or $w(t)$, $w'(t) > 0$ for all $t \in [T - 1, T]$ – see Lemma 3.3.

We thus infer that $w'(T) \leq v'(T)$. But also $w'(t) \leq w'(T)$ for all $t \in [0, T]$ in view of the comment after Lemma 3.3. Consequently

$$w'(t) \leq v'(T)$$

and (4.19) now follows from (4.22). This proves case (i).

(ii) Let $\hat{\omega}$ denote the left hand side of (4.20). We introduce a new operator $\hat{\Phi}$, defined by

$$\hat{\Phi}(\rho) = \begin{cases} \Phi(\rho) & \text{for } 0 \le \rho \le \Phi^{-1}(\hat{\omega}) \\ \dfrac{\hat{\omega}}{\Phi^{-1}(\hat{\omega})}\,\rho & \text{for } \quad \rho \ge \Phi^{-1}(\hat{\omega}). \end{cases} \tag{4.23}$$

Clearly $\hat{\Phi}$ is continuous and increasing on $[0,\infty)$, thus satisfying (A1) and (A2), and moreover $\hat{\Phi}(\infty) = \infty$.

We apply part (i) to problem (4.1), but with Φ replaced by $\hat{\Phi}$. Clearly a solution w exists, and by (4.19) it obeys

$$\|w'\|_\infty \le \hat{\Phi}^{-1}\left(\frac{p_1}{p_0}[\bar{f}(m) + \hat{\Phi}(m)]\right). \tag{4.24}$$

Now from the given assumption (4.20) one finds

$$\Phi(m) \le \frac{p_0}{p_1}\hat{\omega} \le \hat{\omega},$$

since $p_0 \le p_1$, because q is non–increasing. It follows that $m \le \Phi^{-1}(\hat{\omega})$, so $\hat{\Phi}(m) = \Phi(m)$ by (4.23). Therefore (4.24) becomes

$$\|w'\|_\infty \le \hat{\Phi}^{-1}\left(\frac{p_1}{p_0}[\bar{f}(m) + \Phi(m)]\right) = \hat{\Phi}^{-1}(\hat{\omega}) = \Phi^{-1}(\hat{\omega}),$$

again by (4.23). But this is just (4.19) for w, as required. □

Proposition 4.4. *Suppose that* (F2) *is satisfied. Let $q \in C[0,T]$ with $q > 0$ in $[0,T]$, and also assume condition* (3.12) *– or, slightly stronger, that $q \in C^1[0,T)$. Suppose additionally that either $f(u) = 0$ when $u \in (0,\mu)$, $\mu > 0$, or that* (1.6) *holds, that is*

$$\int_0^\delta \frac{ds}{H^{-1}(F(s))} = \infty. \tag{4.25}$$

Then the solution of (4.1) *given in either Proposition* 4.1 *or Proposition* 4.3 *has the properties*

$$w > 0 \quad \text{in } (0,T], \qquad w' > 0 \quad \text{in } [0,T]. \tag{4.26}$$

Proof. Case 1. Let $f(u) = 0$ when $u \in (0,\mu)$. Then from (4.1) we have $[q(t)\Phi(w'(t))]' = 0$ at least for t near 0. Hence in turn $q\Phi \circ w' =$ Constant > 0 for small t (if the constant is zero, then $w' = 0$ for small $t > 0$, and then by continuation for all $t > 0$, which contradicts the boundary condition $w = m$ at $t = T$). Consequently $w'(0) = \Phi^{-1}(\text{Constant}/q(0)) > 0$, so from Lemma 3.3 and the fact that $t_0 = 0$ in the present case, we get $w'(t) > 0$ in $[0,T]$ and $w > 0$ in $(0,T]$ as required.

Case 2. Let (4.25) hold. Note that (3.13) is satisfied in view of (4.3). Also we already know that $w'(0) \ge 0$ and $0 \le w \le m$. In fact, the case $w'(0) = 0$ cannot occur by Proposition 3.6 and assumption (4.25). Consequently $w'(0) > 0$ and the required conclusion then follows as before. □

Remarks. If $(A2)$ is strengthened by adding that q is in $C^1(0,T)$ and $\Phi \in C^1(\mathbb{R}^+)$ with $\Phi' > 0$ in $\mathbb{R}^+$, then one finds easily that the solution w is in $C^2(0,T)$. If also $q' \le 0$, as is frequently the case, then $w'' \ge 0$.

Proposition 4.1 can be improved by allowing a more general version of equation (4.1), namely

$$[q(t)\Phi(w')]' - q(t)B(t,w,w') = 0,$$

provided that B is a continuous function of its variables such that

$$-\kappa\Phi(\rho) \le B(t,u,\rho) \le \kappa\Phi(\rho) + f(u), \qquad |\rho| \le 1,$$

for some constant $\kappa > 0$ and for $f = f(u)$ satisfying the previous assumptions (F1) and (F3). The proof is essentially the same as before, with the exception that the space $X = C[0,T]$ must be replaced by $X = C^1[0,T]$ while the required mapping $\mathcal{H}[w,\sigma]$ is now defined by

$$\mathcal{H}[w,\sigma](t) = \sigma m - \int_t^T \Phi^{-1}\left(\frac{1}{q(s)}\left[\mu_\sigma - \sigma\int_s^T B(\tau, w(\tau), w'(\tau))d\tau\right]\right) ds.$$

This of course makes it more delicate to prove that the mapping is compact, though the argument again follows from the Ascoli–Arzelà theorem. Similarly, proving that any fixed point is uniformly bounded in X takes more effort, but no essentially new or difficult ideas, see [29].

An existence theorem for the problem (4.2) can be given, exactly following the ideas of Proposition 4.1.

Proposition 4.5. *Assume a, h, $q \in C[0,T]$ and $h \geq 0$, $a \geq 0$, $q > 0$ in $[0,T]$. Then problem* (4.2) *with $m \in (0,\overline{\delta})$, and with m and $\int_0^T h(t)\,dt$ suitably small in case $\Phi(\infty) < \infty$, admits a classical distribution solution with the properties $w \in C^1[0,T]$, $w' \geq 0$.*

The proof goes in almost the same way as before for Proposition 4.1, except one must take

$$\mu_1 = q_1[a_1 T\bar{f}(m) + \Phi(m/T)] + \int_0^T h(t)dt, \qquad \text{where} \quad a_1 = \max_{t\in[0,T]} a(t).$$

The question of uniqueness of solutions of (4.1) and (4.2) is also of interest. For this result, we assume the main conditions (A1), (A2), (F1), (F2).

Theorem 4.6. *Assume a, h, $q \in C(0,T)$ and $a \geq 0$, $q > 0$ in $(0,T)$. Then problems* (4.1) *and* (4.2) *admit at most one classical distribution solution with range in* $[0,\delta)$.

Proof. Let w and $\tilde{w}$ be two solutions of (4.2) with ranges in $[0,\delta)$. Then by (4.2) together with (A2) and (F2), we obtain

$$\begin{aligned} 0 &\leq \int_0^T q(t)[\Phi(w'(t)) - \Phi(\tilde{w}'(t))]\cdot[w'(t) - \tilde{w}'(t)]dt \\ &= -\int_0^T a(t)q(t)[f(w(t)) - f(\tilde{w}(t))]\cdot[w(t) - \tilde{w}(t)]dt \leq 0. \end{aligned}$$

It now follows at once that $w \equiv \tilde{w}$ in $[0,T]$ since Φ is strictly increasing. □

It is possible to prove uniqueness with condition (F2) replaced by the weaker hypothesis (F3), when $m < \overline{\delta}$ and q is non–increasing. We omit the discussion, the details being essentially the same as in Theorem 5.3 (*ii*) in the next section.

5. Radial solutions of an exterior Dirichlet problem

In the next section we shall prove the necessity of Theorem 1.2 through the existence of classical solutions of the exterior Dirichlet problem for (1.1), with equality sign. Because of the separate and independent interest of this question, we devote the present section to its consideration.

As in Section 4, we maintain conditions (A1), (A2), (F1). Moreover we consider in place of (F3) the slightly stronger condition

(F3)′ $f(0) = 0$ and f is positive on some interval $(0,\overline{\delta})$, with $\overline{\delta}$ possibly infinite.

Clearly (F3)′ implies (F3), while as noted before (F2) also implies (F3). At the same time (F2) neither implies (F3)′ nor vice versa.

Theorem 5.1. (Exterior Dirichlet Problem). *Assume condition* (F3)$'$ *is satisfied, and let* $\Omega_R = \{x \in \mathbb{R}^n \ : \ |x| > R\}$. *Then for all* $R > 0$ *and* $m \in (0, \bar{\delta})$, *with* m *sufficiently small if* $\Phi(\infty) = \omega < \infty$, *there is a classical radial solution* $u(x) = u(r)$ *of the problem*

$$\operatorname{div}\{A(|Du|)Du\} - f(u) = 0, \qquad u \geq 0 \tag{5.1}$$

in Ω_R, *such that*

$$u(R) = m, \qquad u(x) \to 0 \quad \text{as } |x| \to \infty. \tag{5.2}$$

Moreover $u' < 0$ *whenever* $u > 0$.

The required smallness condition on m when $\omega < \infty$ is given below by (5.3).

Proof. Let $j = 1, 2, \ldots$, $q(t) = (R + j - t)^{n-1}$ and denote by w_j the solution of

$$\begin{cases} [q(t)\Phi(w_t(t))]_t - q(t)f(w(t)) = 0, \\ w(0) = 0, \qquad w(j) = m \in (0, \bar{\delta}), \\ w_t \geq 0 \quad \text{in } [0, j], \end{cases}$$

which exists by Proposition 4.3 and the fact that $q(t)$ is decreasing.

When $\omega < \infty$ we must of course maintain condition (4.20), which in the present case take the form (since $T = j \geq 1$, $p_0 = q(j) = R^{n-1}$, $p_1 = q(j-1) = (R+1)^{n-1}$),

$$\bar{f}(m) + \Phi(m) < \left(\frac{R}{R+1}\right)^{n-1} \omega. \tag{5.3}$$

It follows now that $u_j(r) = w_j(t)$, $t = R + j - r$, is a solution of

$$\begin{cases} [r^{n-1}\Phi(u'(r))]' - r^{n-1}f(u(r)) = 0 \quad ('= d/dr), \\ u(R) = m, \qquad u(R+j) = 0, \\ u' \leq 0 \quad \text{in } [R, R+j] \end{cases}$$

(here recall that Φ is defined for all real ρ, according to the agreement at the beginning of Section 3, namely $\Phi(\rho) = -\Phi(-\rho)$ if $\rho < 0$).

Now by (4.19) we have

$$\|u_j'\|_\infty \leq \Phi^{-1}\left(\left(\frac{R+1}{R}\right)^{n-1} [\bar{f}(m) + \Phi(m)]\right). \tag{5.4}$$

Hence from the Arzelà–Ascoli theorem (and a diagonal process) a subsequence of the functions u_j converges uniformly to a non-negative, non–increasing Lipschitz continuous limit u on every compact subset of $[R, \infty)$.

We shall show that u is the required solution of (5.1), (5.2). Of course $u : [R, \infty) \to [0, m]$, with $u(R) = m$.

In fact u_j satisfies on $[R, R+j]$ the following integral equation corresponding to (4.7),

$$u_j(r) = m - \int_R^r \Phi^{-1}\left(s^{1-n}\left[\mu_j - \int_R^s \tau^{n-1} f(u_j(\tau))d\tau\right]\right) ds.$$

Moreover $u_j'(R) = -\Phi^{-1}(R^{1-n}\mu_j)$, so

$$\mu_j = R^{n-1}\Phi(|u_j'(R)|) > 0.$$

Then by (5.4) we get

$$\mu_j \leq (R+1)^{n-1}[\bar{f}(m) + \Phi(m)].$$

Hence, up to a subsequence, if necessary, the bounded sequence still called $(\mu_j)_j$ must converge to some number $\mu \geq 0$. Letting $j \to \infty$ the limit function u satisfies the integral equation

$$u(r) = m - \int_R^r \Phi^{-1}\left(s^{1-n}\left[\mu - \int_R^s \tau^{n-1} f(u(\tau))d\tau\right]\right) ds. \tag{5.5}$$

But then u is continuous on $[R,\infty)$ by (5.5) and in turn then of class $C^1[R,\infty)$; thus u is also a classical distribution solution of

$$\begin{cases} [r^{n-1}\Phi(u'(r))]' - r^{n-1}f(u(r)) = 0 & \text{in } [R,\infty), \\ u(R) = m; \qquad u \geq 0, \quad u' \leq 0 & \text{in } [R,\infty), \end{cases} \tag{5.6}$$

by (5.5). Of course, the equation on the first line of (5.6) is equivalent to (5.1) for radial functions $u = u(r)$.

To complete the proof of the theorem it therefore remains to show that $u' < 0$ when $u > 0$ and that $u(r) \to 0$ as $r \to \infty$. To obtain the first, note by virtue of (5.6) that should $u' = 0$ at some point r_0 where $u > 0$ then by $(\mathrm{F3})'$ we would have $r^{n-1}\Phi(u'(r)) > 0$ for all $r > r_0$ sufficiently close to r_0, which is absurd.

For the second part, it is first of all the case that u must decrease to some non–negative limit ℓ as $r \to \infty$. Suppose for contradiction that $\ell > 0$. By $(\mathrm{F3})'$ and the fact that $u' < 0$ (since $u > 0$), by integrating (5.6) on $[r, r+1]$, with $R \leq r < \infty$, we get

$$\begin{aligned} \Phi(u'(r+1)) - \left(\frac{r}{r+1}\right)^{n-1} \Phi(u'(r)) &= \frac{1}{(r+1)^{n-1}} \int_r^{r+1} \tau^{n-1} f(w(\tau))\, d\tau \\ &> \left(\frac{r}{r+1}\right)^{n-1} \int_r^{r+1} f(w(\tau))\, d\tau. \end{aligned} \tag{5.7}$$

From $(\mathrm{F3})'$ and the fact that $\ell \leq u \leq \overline{\delta}$ along the solution, one sees that $f(u(r)) > 0$. Hence by (5.6) again, we find that $r^{n-1}\Phi(|u'(r)|)$ is decreasing and in turn also $|u'|$ decreasing. That is, u' is negative and increasing. Consequently one must have $u'(r) \to 0$ as $r \to \infty$. Letting $r \to \infty$ in (5.7) then yields $0 \geq f(\ell) > 0$, which is the required contradiction. □

Theorem 5.2. *Let the hypotheses of Theorem* 5.1 *be satisfied, and suppose also that condition* (F2) *is valid. Then the solution u given by Theorem* 5.1 *is everywhere positive provided that* (1.6) *holds. Conversely if* (1.7) *is satisfied, then u has compact support.*

The proof of the first part of this result will be given following Theorem 1.1 in the next section. Similarly, the proof of the second part of the result will be deferred until after the proof of Theorem 1.2.

Remark. Condition (5.3) is not best possible, and can be improved to the form

$$T_0\bar{f}(m) + \Phi\left(\frac{m}{T_0}\right) \leq \left(\frac{R}{R+T_0}\right)^{n-1} \omega,$$

where $T_0 > 0$ is a positive parameter which can be assigned arbitrarily; this follows easily by redoing Lemma 4.2 and Proposition 4.3 with the respective conditions $T = 1$ and $T \geq 1$ replaced by $T = T_0$ and $T \geq T_0$.

As an example, when $R << 1$ and $A(\rho) = 1/\sqrt{1+\rho^2}$ is the mean curvature operator, with $f(u) = \kappa u$, $\kappa > 0$, and $n = 2$ (equation of a capillary surface under gravity), by taking $T_0 = aR$ with $a >> 1$ we get the solvability condition $m < R$; whereas from (5.3) one gets the weaker condition $m < R/(1+\kappa)$.

An alternative approach to the radial exterior problem, containing a number of precise estimates in the case when $\omega < \infty$ and $\Omega'(0) > 0$, has been given by Turkington [40].

We conclude the section by showing that the solution $u = u(r)$ given in Theorem 5.1 is unique, under various natural conditions. The precise results are as follows.

Theorem 5.3. *Let $m > 0$ and $R > 0$ be fixed.*

(i) Assume $(\mathrm{F3})'$ *is satisfied. Then there cannot be more than one radial solution of* (5.1) *in Ω_R which has a bounded range in $[0,\overline{\delta})$ and satisfies $u(R) = m$. Moreover, any such solution is convex and obeys* (5.2).

(ii) Assume (F3) *is satisfied. Then there cannot be more than one radial solution of* (5.1), (5.2) *in* Ω_R *which has range in* $[0, \overline{\delta})$.

(iii) Assume (F2). *Then there cannot be more than one solution of* (5.1), (5.2) *in* Ω_R, *whether radial or not, which has range in* $[0, \delta)$.

Proof. (i) Let u, v be two solutions of the type described. By the earlier arguments of this section it is evident that u is strictly convex whenever it is positive. Hence $u' \le 0$ for otherwise u would become unbounded for large enough r, contrary to assumption. Then, as in the proof at the end of Theorem 5.1, we get $u(x) \to 0$ as $|x| \to \infty$, that is (5.2) holds. The same of course is true for the solution v.

But then $u \equiv v$ by virtue of Theorem 3.6.7 of [17], when we observe that equation $(*)$ in [17] is exactly (5.1) here, and condition (G1) there (with α replaced by $\overline{\delta}$) is just (F3) here.[5]

(ii) This is again just Theorem 3.6.7 of [17].

(iii) Uniqueness for this case is an immediate consequence of the following comparison result, which we state in a more general form than necessary, in anticipation of later purposes. □

Theorem 5.4. (Weak comparison principle). *Assume* (F2) *is satisfied. Let* u *and* v *be, respectively, classical solutions of* (1.1) *and* (1.2) *in a bounded domain* Ω. *Suppose also that* u *and* v *are continuous in* $\overline{\Omega}$, *with* $v < \delta$ *in* Ω *and* $u \ge v$ *on* $\partial\Omega$. *Then* $u \ge v$ *in* Ω.

The conclusion also holds for exterior domains Ω, *provided that additionally one has*

$$\liminf\{u(x) - v(x)\} \ge 0 \qquad \text{as } |x| \to \infty.$$

Before proving Theorem 5.4 it is convenient to give a simple preliminary lemma

Lemma 5.5. *Let* $\boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$ *be vectors in* $\mathbb{R}^n$. *Then*

$$\{A(|\boldsymbol{\xi}|)\boldsymbol{\xi} - A(|\boldsymbol{\eta}|)\boldsymbol{\eta}\} \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) > 0$$

whenever $\boldsymbol{\xi} \neq \boldsymbol{\eta}$.

Proof. Since $A(\rho) > 0$ when $\rho > 0$ and $\boldsymbol{\xi} \cdot \boldsymbol{\eta} \le |\boldsymbol{\xi}| \cdot |\boldsymbol{\eta}|$, there follows by direct calculation

$$\{A(|\boldsymbol{\xi}|)\boldsymbol{\xi} - A(|\boldsymbol{\eta}|)\boldsymbol{\eta}\} \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \ge \{\Phi(|\boldsymbol{\xi}|) - \Phi(|\boldsymbol{\eta}|)\} (|\boldsymbol{\xi}| - |\boldsymbol{\eta}|)$$

and the conclusion now comes from the strict monotonicity of Φ. □

Proof of Theorem 5.4. We follow the proof of Lemma 3 of [30], first supposing that Ω is bounded.

Let $w = u - v$ in $\overline{\Omega}$. If the conclusion fails, then there exists a point $x_1 \in \Omega$ such that $w(x_1) < 0$. Fix $\varepsilon > 0$ so small that $w(x_1) + \varepsilon < 0$. Consequently, since $w \ge 0$ on $\partial\Omega$ it follows that the function $w_\varepsilon = \min\{w + \varepsilon,\ 0\}$ is non–positive and has compact support in Ω. By the distribution meaning of solutions, taking the Lipschitzian function w_ε as test function, we get

$$\int_\Omega \{A(|Du|)Du - A(|Dv|)Dv\} Dw_\varepsilon \le \int_\Omega \{f(v) - f(u)\} w_\varepsilon. \tag{5.8}$$

The left hand side of (5.8) is positive due to Lemma 5.5 and the fact that $Dw_\varepsilon \equiv Dw = Du - Dv \not\equiv \mathbf{0}$ when $w + \varepsilon < 0$, while otherwise $Dw_\varepsilon = \mathbf{0}$ (a.e.).

Moreover, when $w + \varepsilon < 0$ there holds $0 \le u < v - \varepsilon < \delta$; hence $f(v) - f(u) \ge 0$ since $f(s)$ is non–decreasing for $s < \delta$ by (F2). Thus the right hand side of (5.8) is non–positive, a contradiction.

The case when Ω is an exterior domain is proved in almost exactly the same way. We leave the details to the reader. □

[5]The proof of Theorem 3.6.7 in [17] relies on the preceding Theorems 3.6.1 – 3.6.5. All of these results are straightforward, except possibly for Theorem 3.6.5. A simpler proof of the latter result can however be given, using the ideas of Theorem 3.6.4.

Theorem 5.4 is closely related to Theorem 10.1 of [18], and equally does not require differentiability conditions for the nonlinear terms; see also Theorem 10.5.

6. Proofs of Theorems 1.1 and 1.2

With the work of the preceding two sections available, we can now turn to the main results of the paper, proofs of the Strong Maximum Principle, Theorem 1.1, and the Compact Support Principle, Theorem 1.2.

Proof of Theorem 1.1. We recall that Φ is defined for $\rho < 0$ by $\Phi(\rho) = -\Phi(-\rho)$.

The radial function $v(x) = w(t)$, $t = R - r$, $r = |x|$, where w is given by Proposition 4.1 with $m < \delta$, $q(t) = (R-t)^{n-1}$ and $T = R/2$, satisfies the differential equation (1.2) in the annular set $E_R = \{x \in \mathbb{R}^n \ : \ R/2 \le |x| \le R\}$. Writing $\ ' = d/dt = -d/dr$ in accordance with Proposition 4.1, one has $Dv(x) = -w'(t)\boldsymbol{x}/r$ for $R/2 \le |x| \le R$. Moreover $w'(t) > 0$ for $t \in [0, R/2]$ by Proposition 4.4 and the fact that $\Phi(w')$ is continuously differentiable (see Proposition 4.1 (i)). Hence we find

$$\begin{aligned} \operatorname{div}\{A(|Dv|)Dv\} - f(v) &= - \operatorname{div}\{A(w')w'x/r\} - f(w) \\ &= [\Phi(w')]' - \frac{(n-1)}{r}\Phi(w') - f(w) \\ &= \frac{1}{q(t)}[q(t)\Phi(w')]' - f(w) = 0, \end{aligned} \tag{6.1}$$

where at the second step we use $D(A(w')w') = -[\Phi(w')]'\boldsymbol{x}/r$. Of course one has $Dv(x) = -w'(R-r)\boldsymbol{x}/r \neq \mathbf{0}$ in $[R/2, R]$.

This being shown, the proof of sufficiency is now exactly the same as in the standard demonstration of the strong maximum principle for linear equations (see e.g. the proof of Theorem 3.5 on page 35 in [18]); here one uses the fact that the comparison function v constructed above satisfies the following conditions, see the proof of Lemma 3.4 on page 34 in [18]:

(i) $v > 0$ in $[R/2, R)$ by Proposition 4.4,
(ii) $v = 0$ when $|x| = R$ by Proposition 4.1,
(iii) $\partial v/\partial \boldsymbol{n} = v' < 0$ when $|x| = R$, where $\boldsymbol{n}$ is the outer normal to ∂E_R,
(iv) $v = m$ when $|x| = R/2$ by Proposition 4.1,

where m, $R > 0$ can be taken arbitrarily small and the origin of coordinates can be chosen arbitrarily in Ω. Note that the use of the weak maximum principle (Corollary 3.2 of [18]) is here replaced by application of Theorem 5.4. This completes the proof of the sufficiency part of Theorem 1.1.

As remarked in the introduction, the necessity is due to Diaz [11]. Hence Theorem 1.1 is proved (see also comment **4** at the end of the section and the further remarks at the end of Section 7). ☐

Proof of first part of Theorem 5.2. Because of (1.6) the strong maximum principle is valid for (1.1). But since $u(R) = m > 0$ and because u is a non–negative (radial) solution of (1.1), it now follows that $u > 0$ on the entire domain of the solution. ☐

Proof of Theorem 1.2. To prove necessity, suppose (1.7) fails, that is (1.6) holds. By Theorem 5.1 and the *first part* of Theorem 5.2, therefore, there exists a *positive* classical solution u of (1.1) with equality sign (and thus also of (1.2) with equality), in the domain $\Omega_R = \{x \in \mathbb{R}^n \ : \ |x| > R\}$, such that $u(x) \to 0$ as $|x| \to \infty$. This violates the compact support principle. Hence (1.7) is necessary.

For the sufficiency we follow the proof of Theorem 2 of [30]. By (1.7) we can define

$$C = \int_0^\delta \frac{ds}{H^{-1}(F(s))} < \infty, \tag{6.2}$$

where, if necessary, one can take $\delta > 0$ smaller so that $F(\delta) < H(\infty)$. Introduce $w = w(r)$, $0 \le r \le C$, by

$$r = \int_{w(r)}^{\delta} \frac{ds}{H^{-1}(F(s))}. \tag{6.3}$$

Differentiation gives

$$-\frac{w'(r)}{H^{-1}(F(w(r))} = 1 \qquad \text{for } 0 \le r \le C,$$

that is, w is of class $C^1[0, C]$, with $w(0) = \delta$, $w(C) = 0$, $0 \le w \le \delta$, and $w'(r) < 0$ for $0 \le r < C$. Also $H(|w'|) = F(w)$, so $H(|w'|)$ is of class $C^1[0, C]$ with $[H(|w'|)]' = f(w)w'$. Then from Lemma 3.1 (ii) with $T = C$, we see that $\Phi(|w'|)$ is of class $C^1(0, C)$ and

$$-[\Phi(|w'|)]' = f(w) \quad \text{for } 0 < r < C. \tag{6.4}$$

Obviously $w(r) \to 0$, $w'(r) \to 0$ and $[\Phi(|w'|)]' \to 0$ as $r \to C$. Therefore, by defining $w(r) \equiv 0$ for $r \ge C$, it is clear that w becomes a C^1 solution of (6.4) in $(0, \infty)$.

Now let u be the solution of (1.2) in an exterior domain Ω with $u(x) \to 0$ as $|x| \to \infty$. We must show that u has compact support in Ω. To begin with, clearly there exists $R_0 \ge R$ such that $u(x) < \delta$ if $|x| \ge R_0$. For any $x \in \Omega_0 = \{x \in \mathbb{R}^n : |x| > R_0\}$, define $v(x) = w(|x| - R_0)$. Consequently, for $x \in \Omega_0$, and $r = |x|$, we have

$$\operatorname{div}\{A(|Dv|)Dv\} - f(v) = -[\Phi(|v'|)]' + \frac{(n-1)}{r}\Phi(v') - f(v) \le 0 \tag{6.5}$$

in view of (6.4) (which now holds in $(0, \infty)$), and the fact that $\Phi(v') \le 0$ when $v' \le 0$. Since $0 \le u(x) < \delta = v(x)$ on $\partial\Omega_0$, and since $u(x)$, $v(x) \to 0$ as $|x| \to \infty$, we can apply the comparison Theorem 5.4 (with the roles of u and v interchanged) to obtain $0 \le u(x) \le v(x)$ in Ω_0. In particular $u(x) = 0$ when $|x| \ge R_1 = R_0 + C$, as required. □

Proof of second part of Theorem 5.2. Recall that (F3) holds by hypothesis. Then because of (1.7) the compact support principle Theorem 1.2 is valid for equation (5.1). But since u is a non–negative (radial) solution of (5.1) with $u(x) \to 0$ as $|x| \to \infty$, it now follows that u has compact support in the domain $|x| \ge R$. □

Remarks. **1.** The sufficiency part of Theorem 1.2 is closely related to Theorem 4 of [31], by specializing the results there to the matrix $a_{ij} = A(|\boldsymbol{\xi}|)\delta_{ij} + [A'(|\boldsymbol{\xi}|)/|\boldsymbol{\xi}|]\xi_i\xi_j$ which arises by expansion of the divergence term in (1.2). This specialization requires, however, two assumptions which are not needed here, first that the operator A be of class $C^1(0, \infty)$, and second, that the solutions in consideration should be of class C^2 at points of Ω where $Du \ne \mathbf{0}$. In the proof of Theorem 4 of [31] it is not evident that an appropriate comparison principle can be applied without the further assumption that the nonlinearity f be non–decreasing for small $u > 0$ – that is, for the validity of Theorem 4 of [31] this additional assumption, which is exactly (F2) above, seems to be required as well. For the special case of the degenerate Laplacian, see also [13].

The proof of sufficiency we have given is in fact not different in its underlying ideas from those in [4], [6], [13], [31], [41], the principal improvements here being the direct approach, the generality of the solution class, and the clarification of the method.

We note also that Diaz, Saa and Thiel have stated a version of Theorem 1.1, see Theorem 6 of [14], but with insufficient proof.

2. The last sentence of the proof of Theorem 1.2 gives an a priori estimate for the support of the solution u.

3. Theorem 1.2 also applies when f satisfies the alternative conditions:

(f1) $f \in C(0, \infty)$,
(f2) f is a maximal graph with $f(0) = 0$ and $\liminf_{u \to 0} f(u) > 0$ (or $+\infty$);

rather than (F1), (F2). We can transform the vertical segment of f at $u = 0$ into a linear segment with finite slope, thus arriving at a function $\bar{f} \leq f$ satisfying (F1) and (F2). But then every solution of (1.2) remains a solution of (1.2) with f replaced by $\bar{f}$, and the result of Theorem 1.2 continues to apply. A similar argument can be used also for maximal monotone graphs f, see [41].

4. *Another proof of the necessity of* (1.6) *for the Strong Maximum Principle.* Suppose $f(u) > 0$ for $u > 0$ and that (1.6) fails, that is (1.7) holds. We can then introduce the function $w = w(r)$, defined on $[0, \infty)$, as in the proof of Theorem 1.2. For any $x \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, let $u(x) = w(x_n)$. By (6.4), u is obviously a solution of (1.1), *with the equality sign*, in the domain $\Omega = R^n_+$. Clearly $u(0, \ldots, 0, C) = w(C) = 0$ and at the same time $u \not\equiv 0$ in Ω. Hence the strong maximum principle fails. □

5. The necessity of condition (1.6) for the Strong Maximum Principle can be obtained under a weaker hypothesis than (F2). In fact, it is enough to replace (F2) by

$$\text{(F2)}' \qquad f(0) = 0 \quad \text{and} \quad F(s) > 0 \quad \text{for } s \in (0, \delta).$$

This is because the principal construction required for Diaz' proof uses only condition (F2)$'$; see also the construction of the function $w = w(r)$ noted just above.

6. The necessity also yields a direct and simple counterexample to the unique continuation question for the equation $\operatorname{div}\{A(|Du|)Du\} - f(u) = 0$, when (1.7) holds. That is, the function $u(x) = w(x_n)$ shows that a solution in a domain Ω may vanish in a subdomain without vanishing throughout Ω. Theorems 7.2 and 7.5 below give more sophisticated counterexamples.

7. Dead cores

An elliptic equation or inequality is said to have a *dead core solution* u in some domain $\Omega \subset \mathbb{R}^n$ provided that there exists an open subset Ω_1 with compact closure in Ω such that

$$u \equiv 0 \quad \text{in } \Omega_1, \qquad u > 0 \quad \text{in } \Omega \setminus \overline{\Omega_1}.$$

The condition $u > 0$ could be replaced by $u \neq 0$, but for definiteness (and physical reality) we prefer the condition as stated.

In what follows we maintain the original conditions (A1), (A2), (F1), (F2), unless otherwise stated. The additional condition

$$\text{(7.1)} \qquad f \text{ is positive in } (0, \delta)$$

will also be important.

Lemma 7.1. (Dead core lemma). *Suppose* (7.1) *and* (1.7) *are satisfied. For fixed σ in $(0, 1)$, define*

$$\text{(7.2)} \qquad C_\sigma = \int_0^\delta \frac{ds}{H^{-1}(\sigma F(s))} \quad (> 0).$$

Then for every $C \in (0, C_\sigma)$ there exists a number $\gamma = \gamma(C) \in (0, \delta)$ and a function $w \in C^1[0, C]$ such that

(i) $$\gamma \to 0 \quad \textit{as } C \to 0,$$

(ii) $$w(0) = w'(0) = 0, \quad w(C) = \gamma; \qquad 0 \leq w' \leq H^{-1}(F(\gamma)),$$

(iii) $$[\Phi(w'(t))]' = \sigma f(w(t)) \quad \textit{for } t \in (0, C),$$

(iv) $$\Phi(w'(t)) \leq \sigma t f(w(t)) \quad \textit{for } t \in (0, C).$$

[Here we can assume without loss of generality that $\sigma F(\delta) < H(\infty)$.]

Proof. First note that the integral in (7.2) is convergent, in view of Lemma 3.2 and (1.7).

For given $C \in (0, C_\sigma)$, we take $\gamma \in (0,\delta)$ so that

$$0 < C = \int_0^\gamma \frac{ds}{H^{-1}(\sigma F(s))};$$

clearly $\gamma = \gamma(C)$ is uniquely determined by C, and of course $\gamma \to 0$ as $C \to 0$.

Now define $w : [0, C] \to \mathbb{R}$ by

$$t = \int_0^{w(t)} \frac{ds}{H^{-1}(\sigma F(s))}.$$

Hence

$$\frac{w'(t)}{H^{-1}(\sigma F(w(t)))} = 1,$$

that is $H(w') = \sigma F(w)$ and in turn $[H(w')]' = \sigma f(w)w'$. Obviously part *(ii)* of the Lemma is satisfied; moreover, since $w' > 0$ on $(0, C]$, from Lemma 3.1(ii) we obtain part *(iii)*.

An integration using parts *(ii)*, *(iii)* and (F2) shows that also $\Phi(w'(t)) \le \sigma t f(w(t))$; see the proof of Lemma 3.4. This completes the proof. □

Theorem 7.2. *Suppose* (7.1) *and* (1.7) *are satisfied. Let $R > 0$ be fixed. Then the equation*

$$\operatorname{div}\{A(|Du|)Du\} - f(u) = 0 \tag{7.3}$$

admits a non–negative dead core solution in B_R.

Proof. Fix $\sigma = 1/n$. Take $0 < C < \min\{C_\sigma,\ R\}$ and put $S = R - C$. Define the radial function $v(r) = w(r - S)$, $r = |x| \in [S, R]$, where $\gamma = \gamma(C)$ and $w(t)$ are as given in Lemma 7.1. Then for $r \in (S, R)$

$$\begin{aligned}\operatorname{div}\{A(|Dv(x)|)Dv(x)\} - f(v(x)) &= [\Phi(v'(r))]' + \frac{n-1}{r}\Phi(v'(r)) - f(v(r))\\ &\le \left\{\sigma\left[1 + (n-1)\frac{r-S}{r}\right] - 1\right\} f(v(r))\\ &\le (\sigma n - 1) f(v(r)) = 0,\end{aligned} \tag{7.4}$$

where we have used parts *(iii)* and *(iv)* of Lemma 7.1, and the fact that $f(v(r)) > 0$ since $v((S, R]) \subset (0, \delta)$. Of course also

$$v(S) = v'(S) = 0, \qquad v(R) = \gamma < \delta.$$

Consider the radial solution $u = u(r)$, $r = |x|$, of the problem

$$\begin{cases}\operatorname{div}\{A(|Du|)Du\} - f(u) = 0,\\ u(S) = 0, \quad u(R) = m > 0,\end{cases}$$

given by Proposition 4.1, with $q(r) = r^{n-1}$ and with $m \in (0, \gamma)$ suitably small (translate coordinates by $r = t + S$ and take $T = R - S = C$). Also suppose (4.5) is obeyed if $\Phi(\infty) < \infty$.

Now apply Theorem 5.4, with the roles of u and v interchanged. This gives $0 \le u(r) \le v(r)$, $r \in [S, R]$. Hence $u'(S) = 0$ since $v'(S) = 0$. Therefore u can be extended as a solution of (7.3) to the entire set B_R by putting $u \equiv 0$ in B_S. This proves the existence of the required dead core solution of (7.3). □

Theorem 7.3. *Suppose* (7.1) *and* (1.7) *are satisfied. Let $R > 0$ be fixed. Then any solution u of* (1.2) *in B_R with range in $[0, \delta)$ and for which $u(x)$ is suitably small on ∂B_R, is a dead core solution.*

This is shown in the same way as Theorem 7.2.

Corollary 7.4. *Suppose condition* (F2) *is replaced by the assumption that f is non–decreasing in $(-\delta,\delta)$. Assume also that $uf(u)>0$ for $u\neq 0$ and that* (1.7) *holds for both ranges $(0,\delta)$ and $(-\delta,0)$.*

Let u be a solution of

$$[\text{sign } u(x)]\cdot[\text{div}\{A(|Du|)Du\}-f(u(x))]\geq 0$$

in B_R with range in $(-\delta,\delta)$. Then u vanishes in B_S for some $S\in(0,R)$, provided $|u(x)|$ is suitably small on ∂B_R.

For p–regular equations (see Section 11), and therefore in particular without monotonicity conditions, this result was obtained by Diaz and Veron [15].

Lemma 7.1 gives a companion result to Proposition 4.4. Namely, let (7.1) and (1.7) be satisfied. Then if m is suitably small the solution of (4.1) given by Proposition 4.1 has the property $w'(0)=0$. The proof is obvious, after what has gone before.

We conclude by noting the *existence* of compact support solutions of equation (1.2), with the equality sign. In fact, one can interpret a compact support solution as a dead core *at infinity*.

Theorem 7.5. *Suppose* (7.1) *and* (1.7) *are satisfied. Let $R>0$ be fixed. Then* (7.3) *admits a (non–trivial) non–negative compact support solution in $\Omega_R=\{x\in\mathbb{R}^n : |x|>R\}$.*

This is just the second part of Theorem 5.2. A related result for the p–Laplace operator is well–known, see [13].

Of course, if (1.7) fails, the strong maximum principle shows that a non–negative compact support solution would in fact vanish identically.

A dead core with bursts. It is known that when (7.1) and (1.7) hold and when f appropriately changes sign for $u>\delta$, there are non–negative radially symmetric solutions v of (7.3) having compact support; see for example [17]. Let R_1 be the support radius of such a solution. Next choose R and S in Theorem 7.2 so that $S>>R_1$, and let w denote the corresponding dead core solution. This being done, we can now replace the solution w on the set B_{R_1}, where it vanishes, by the solution v, thus obtaining a new solution u which is then positive in B_{R_1} and $B_R\setminus B_S$, and otherwise vanishes. This solution may be considered as a dead core with a symmetric *burst* centered at the origin.

Of course, the same procedure may be repeated at other suitably chosen origins in B_S, giving rise to multiple bursts. Naturally a given ball B_S can accomodate only a certain number of bursts, but the larger are R and S the more bursts which can be allowed.

Remarks. The existence of a dead core in Theorem 7.2 supplies still another counterexample to the strong maximum principle when (1.7) holds. It is worth pointing out here that this counterexample is in fact a *solution of equation* (7.3); that is one proves in this way a sharper version of the necessity of condition (1.6) for the strong maximum principle.

The results of Theorems 7.2 and 7.3 can be extended to more general quasilinear cases, as anticipated in the Remark at the end of Section 4. See the forthcoming paper [29].

We wish to thank Professor L.A. Peletier for helpful discussions concerning the material of this section.

8. More general quasilinear inequalities

Let $\mathcal{D}$ be a domain in $\mathbb{R}^n$. Let $[a_{ij}(x,u)]$, $i,j=1,\cdots,n$, be a continuously differentiable, symmetric coefficient matrix defined for $x\in\mathcal{D}$, $u\geq 0$, and which is positive definite in these variables, namely

$$a_{ij}(x,u)\eta_i\eta_j>0,\qquad \boldsymbol{\eta}\in\mathbb{R}^n\setminus\{\mathbf{0}\}. \tag{8.1}$$

We shall suppose furthermore that the principal operator $A=A(\rho)$ satisfies the following strengthened versions of (A1), (A2), namely

(A1)′ $A\in C^1(0,\infty)$,

(A2)′ $\Phi'(\rho) > 0$ for $\rho > 0$, and $\Phi(\rho) \to 0$ as $\rho \to 0$.

8.1. **The strong maximum principle**

Consider the differential inequality

$$D_i\{a_{ij}(x,u)A(|Du|)D_j u\} - B(x,u,Du) \le 0, \qquad u \ge 0, \tag{8.2}$$

in a domain $\Omega \subset \mathcal{D}$.

We shall treat the following main conditions on the (continuous) function $B(x,u,\boldsymbol{\xi})$:

(B1) $B(x,u,\boldsymbol{\xi}) \le \kappa\Phi(|\boldsymbol{\xi}|) + f(u)$,
(B2) $B(x,u,\boldsymbol{\xi}) \ge -\kappa\Phi(|\boldsymbol{\xi}|) + g(u)$

for $x \in \Omega$, $u \ge 0$, and all $\boldsymbol{\xi} \in \mathbb{R}^n$ with $|\boldsymbol{\xi}| \le 1$, where $\kappa > 0$ and the nonlinearities f, g obey (F1) and (F2).

It is interesting to observe that for the validity of the following results the function $B(x,u,\boldsymbol{\xi})$ *need not* be non–decreasing in the variable u! This corresponds to the situation of Theorem 2.2 where the coefficient $c(x)$ is *not* required to satisfy a sign condition for the validity of the conclusion. (For a statement of the strong maximum principle, see the second paragraph preceding Theorem 1.1.)

Theorem 8.1. (Strong maximum principle). *Assume* (B1). *For the strong maximum principle to hold for* (8.2) *it is sufficient that either $f \equiv 0$ in $[0,\mu)$, $\mu > 0$, or that* (1.6) *is satisfied.*

Assume (B2). *For the strong maximum principle to hold for* (8.2) *it is necessary that either $g \equiv 0$ for $u \in [0,\mu)$, $\mu > 0$, or that*

$$\int_0^\delta \frac{ds}{H^{-1}(G(s))} = \infty \tag{8.3}$$

holds, where $G(u) = \int_0^u g(s)ds$.

The sufficiency was obtained in Theorem 1′ of [30] under the additional technical assumption (2.5) of [30], and in Theorem 3 of [27] without the assumption (2.5) of [30]. In both papers, moreover, the matrix a_{ij} was assumed to be *independent of the variable u.* For other comments on earlier work, see the Introduction and also Section 4 of [30].

Proof. Sufficiency. We follow the proof of Theorem 3 of [27], using however a modified version of the auxiliary function constructed in Proposition 4.1.

We first introduce the modified coefficient matrix

$$\hat{a}_{ij}(x) \equiv a_{ij}(x,u(x)),$$

obviously still continuously differentiable in Ω. Let O be an arbitrary origin in Ω. Put $E_R = \{x \in \mathbb{R}^n \ : \ R/2 \le |x| \le R\}$ where R is supposed sufficiently small that E_R is in Ω. Define

$$\lambda = \text{min eigenvalue of } [\hat{a}_{ij}(x)] \text{ in } E_R = \text{min eigenvalue of } [a_{ij}(x,u(x))] \text{ in } E_R,$$

$$\Lambda = \text{max eigenvalue of } [\hat{a}_{ij}(x)] \text{ in } E_R = \text{max eigenvalue of } [a_{ij}(x,u(x))] \text{ in } E_R,$$

and let α be a constant such that

$$|\xi_j D_i\, \hat{a}_{ij}(x)| \le \alpha|\boldsymbol{\xi}|$$

for all $x \in E_R$ and $\boldsymbol{\xi} \in \mathbb{R}^n$. Clearly such a constant α exists since $u \in C^1(\Omega)$ and E_R is a compact subset of Ω. It is easy to see that

$$D_i\left(\hat{a}_{ij}(x)\frac{x_j}{r}\right) = (D_i\,\hat{a}_{ij}(x))\,\frac{x_j}{r} + \frac{\hat{a}_{ij}(x)}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right),$$

so for $x \in E_R$,

$$\left\| D_i\left(\hat{a}_{ij}(x)\frac{x_j}{r}\right)\right\| \le \alpha + \frac{n-1}{r}\Lambda. \tag{8.4}$$

Define
$$k = \frac{(n-1)\Lambda + (\alpha+\kappa)R}{\lambda}.$$
We can now introduce the radial Hopf–type comparison function $v(x) = w(t)$, $t = R - r$, $r = |x|$, where w is the unique solution (see Theorem 4.6) of (4.1), given by Proposition 4.1 when $m < \delta$, $q(t) = (R-t)^k$, $T = R/2$ and f is replaced by f/λ. Moreover, since
$$\int_0^\delta \frac{ds}{H^{-1}(\lambda^{-1}F(s))} = \infty$$
by Lemma 3.2 and (1.6), one sees that Proposition 4.4 applies to the solution w.

Thus $Dv(x) = -w'(R-r)\boldsymbol{x}/r \neq \mathbf{0}$ in E_R. Also, by restricting m to be even smaller if necessary – see Proposition 4.1 – one can maintain
$$0 < |Dv| < 1. \tag{8.5}$$
Now we can carry out the following crucial calculation:
$$\begin{aligned} &D_i\{\hat{a}_{ij}(x)A(|Dv|)D_jv\} - \kappa\Phi(|Dv|) - f(v) \\ &\qquad = \hat{a}_{ij}(x)\frac{x_ix_j}{r^2}[\Phi(w')]' - D_i\left\{\hat{a}_{ij}(x)\frac{x_j}{r}\right\}\Phi(w') - \kappa\Phi(w') - f(w) \\ &\qquad \geq \hat{a}_{ij}(x)\frac{x_ix_j}{r^2}\left\{[\Phi(w')]' - \frac{k}{r}\Phi(w') - \frac{f(w)}{\lambda}\right\} \\ &\qquad = \hat{a}_{ij}(x)\frac{x_ix_j}{r^2}\left\{\frac{1}{q(t)}[q(t)\Phi(w')]' - \frac{f(w)}{\lambda}\right\} = 0 \end{aligned} \tag{8.6}$$
by construction of w, that is
$$D_i\{\hat{a}_{ij}(x)A(|Dv|)D_jv\} - \kappa\Phi(|Dv|) - f(v) \geq 0 \tag{8.7}$$
in E_R, with
$$v \geq 0, \quad 0 < |Dv| < 1; \qquad v(R/2) = m, \quad v(R) = 0.$$
We next require a comparison result corresponding to Theorem 5.4, but applying to the more general inequality (8.2).

Lemma 8.2. (Comparison lemma). *Let u and v be respectively solutions of* (8.2) *and* (8.7) *in a bounded domain Ω, and let* (B1) *be satisfied. Suppose that u and v are continuous in $\overline{\Omega}$; and that*
$$0 \leq v < \delta, \quad 0 < |Dv| < 1 \quad \text{in } \Omega; \qquad u \geq v \quad \text{on } \partial\Omega.$$
Then $u \geq v$ in Ω.

Proof. By (8.7) we have
$$D_i\{a_{ij}(x,u(x))A(|Dv|)D_jv\} - \kappa\Phi(|Dv|) - f(v) \geq 0, \qquad 0 \leq v < \delta, \quad |Dv| < 1,$$
in Ω, while from (8.2) and (B1),
$$D_i\{a_{ij}(x,u(x))A(|Du|)D_ju\} - \kappa\Phi(|Du|) - f(u) \leq 0, \qquad u \geq 0,$$
this being valid of course only when $|Du| \leq 1$.

In turn, since $|Du| + |Dv| \geq |Dv| > 0$, we can apply Theorem 10.1 (together with the remark after Corollary 10.4). In particular, Lemma 8.2 follows from the identifications $a = 0$, $b = 1$, and
$$\hat{A}_i(x,\boldsymbol{\xi}) = A(|\boldsymbol{\xi}|)a_{ik}(x,u(x))\xi_k; \qquad \hat{B}(x,z,\boldsymbol{\xi}) = \kappa\Phi(|\boldsymbol{\xi}|) + f(z), \qquad |\boldsymbol{\xi}| \leq 1$$
provided we show that the matrix $[D_{\xi_j}\hat{A}^i(x,\boldsymbol{\xi})]$ is positive definite for $\boldsymbol{\xi} \neq \mathbf{0}$. But
$$D_{\xi_j}\hat{A}^i(x,\boldsymbol{\xi}) = a_{ik}(x,u(x))b_{kj}(\boldsymbol{\xi}),$$
where
$$b_{kj}(\boldsymbol{\xi}) = A(|\boldsymbol{\xi}|)\delta_{kj} + \frac{A'(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\,\xi_k\xi_j, \qquad \boldsymbol{\xi} \neq \mathbf{0}.$$

The matrix $[b_{kj}(\boldsymbol{\xi})]$ has eigenvalues $A(|\boldsymbol{\xi}|)$ (repeated $n-1$ times) and $\Phi'(|\boldsymbol{\xi}|)$. By assumption (A2)$'$ we have $\Phi'(|\boldsymbol{\xi}|) > 0$ for $\boldsymbol{\xi} \neq \mathbf{0}$, while also

$$A(|\boldsymbol{\xi}|) = \Phi(|\boldsymbol{\xi}|)/|\boldsymbol{\xi}| > 0, \qquad \text{for } \boldsymbol{\xi} \neq \mathbf{0},$$

again by (A2)$'$. Hence $[b_{ij}]$ is positive definite for $\boldsymbol{\xi} \neq \mathbf{0}$. Because $[a_{ij}(x,u)]$ is assumed positive definite, it now follows that $[D_{\xi_j} \hat{A}_i(x,\boldsymbol{\xi})]$ is positive definite for $x \in \Omega$ and $\boldsymbol{\xi} \neq \mathbf{0}$, completing the proof. □

The point of Lemma 8.2 is that if $|Dv| > 0$ in Ω, then just as for Theorem 5.4 *it is not necessary to have ellipticity at the value* $\boldsymbol{\xi} = \mathbf{0}$. We remark that it is exactly in the application of this lemma that the strengthened condition (A2)$'$ is needed.

The rest of the proof of sufficiency in Theorem 8.1 is now essentially the same as the sufficiency part of Theorem 1.1. The main change is that at the last step we rely on Lemma 8.2 instead of Theorem 5.4.

Necessity. This follows the corresponding arguments in Theorem 1.1. It is necessary to exhibit, for each x_0 in $\mathcal{D}$, a domain Ω in $\mathcal{D}$ with x_0 in Ω, and a solution v of (8.2) in Ω such that $v(x_0) = 0$ but $v \not\equiv 0$ in Ω.

The assumption to be made for this purpose is that (B2) holds, with $g(u) > 0$ for $u > 0$, together with the negation of (8.3), namely

$$\int_0^\delta \frac{ds}{H^{-1}(G(s))} < \infty. \tag{8.8}$$

Choose $R < 1$ so small that the closure of the domain $\Omega = B_R(x_0)$ is in $\mathcal{D}$. As at the beginning of the proof, let

$$\lambda = \text{min eigenvalue of } [a_{ij}(x,z)] \text{ in } \Omega, \qquad \Lambda = \text{max eigenvalue of } [a_{ij}(x,z)] \text{ in } \Omega$$

for all values $0 \le z \le \delta$. Also let α be such that

$$|\xi_j D_i\, a_{ij}(x,u(x))| \le \alpha |\boldsymbol{\xi}|$$

when $x \in \Omega$, $\boldsymbol{\xi} \in \mathbb{R}^n$, $0 \le u(x) \le \delta$, $|Du| \le b = H^{-1}(G(\delta))$. As before, clearly such a value a can be found. Finally, define

$$\sigma = (n\Lambda + \alpha + \kappa)^{-1},$$

where κ is given by (B2).

Consider the dead core function $v(r) = w(r - S)$, $S \le r \le R$, $r = |x - x_0|$, given in Theorem 7.2 (and using the notation there), but constructed with the function f replaced instead by g and with the new value of σ given above. Clearly v can be extended as a C^1 function to all of Ω by putting $v \equiv 0$ for $0 \le r < S$.

Then we find, see (7.4),

$$\begin{aligned} &D_i\{a_{ij}(x,v(x))A(|Dv|)D_j v\} - B(x,v(x),Dv(x)) \\ &\qquad \le D_i\{a_{ij}(x,v(x))A(|Dv|)D_j v\} + \kappa\Phi(|v'|) - g(v) \qquad \text{by (B2)} \\ &\qquad \le a_{ij}(x,v(x))\frac{x_i x_j}{r^2}[\Phi(|v'|)]' + \left(\alpha + \kappa + \Lambda\frac{n-1}{r}\right)\Phi(|v'|) - g(v) \\ &\qquad \le \Lambda\sigma g(v) + \left(\alpha + \kappa + \Lambda\frac{n-1}{S}\right)C\sigma g(v) - g(v) \\ &\qquad \le [\sigma(n\Lambda + \alpha + \kappa) - 1]g(v) = 0; \end{aligned} \tag{8.9}$$

in obtaining (8.9), note first that when $r = |x - x_0| < S$ there is nothing to show since $v \equiv 0$; on the other hand, for $r \ge S$ we apply the estimates of Lemma 7.1 in the same way as in previous proofs, together with the relations $0 < C < R \le 1$ and $0 < C \le S$; see the proof of Theorem 7.2. Since v has the dead core $B_S(x_0)$, and is otherwise positive in $\Omega = B_R$, the proof is complete. □

Corollary 8.3. *Assume that both* (B1) *and* (B2) *are satisfied, and that there exists* $c > 0$ *such that* $g(u) \geq cf(u)$ *for* $u \in [0, \delta]$*. Then the strong maximum principle holds for* (8.2) *if and only if either* $f \equiv 0$ *in* $[0, \mu]$*,* $\mu > 0$*, or* (1.6) *holds.*

We close the section with the following useful boundary point lemma, which will be required for the proof of Theorem 8.5 below.

Corollary 8.4. (Boundary point lemma). *Let* $x_0 \in \partial\Omega$ *and suppose that* Ω *satisfies an interior sphere condition at* x_0*.*

Let u *be a* C^1 *solution of* (8.2) *in* $\overline{\Omega}$*, with* $u > 0$ *in* Ω *and* $u = 0$ *at* x_0*. Assume that* (B1) *holds and that either* $f \equiv 0$ *in* $[0, \mu)$*,* $\mu > 0$*, or that* (1.6) *is satisfied. Then* $\partial u/\partial \boldsymbol{n} < 0$ *at* x_0*, where* $\boldsymbol{n}$ *is the outer normal to* $\partial\Omega$ *at* x_0*.*

Proof. By the interior sphere condition there exist $y \in \Omega$ and $R > 0$ such that the open ball $B_R(y) \subset \Omega$ and $x_0 \in \partial B$. Let v be the solution of (8.7) given in Theorem 8.1 and put $\tilde{u}(x) = v(|x - y|)$. Then as from Lemma 8.2 it follows that

$$u(x) \geq \tilde{u}(x) \quad \text{in } B_R(y) \setminus B_{R/2}(y)$$

provided that $m > 0$ is sufficiently small. This completes the proof, since $\partial\tilde{u}/\partial\boldsymbol{n} = v'(R) < 0$. □

8.2. **The compact support principle**

There is a corresponding compact support principle for the reversed inequality

$$D_i\{a_{ij}(x, u(x))A(|Du|)D_j u\} - B(x, u, Du) \geq 0, \qquad u \geq 0, \qquad x \in \Omega, \tag{8.10}$$

where Ω is unbounded, with $\Omega_R = \{x \in \mathbb{R}^n \,:\, |x| > R\} \subset \Omega \subset \mathcal{D}$ for some $R > 0$. (For the statement of the compact support principle, see the first paragraph before Theorem 1.2 in the Introduction.)

The conditions on the matrix $a_{ij}(x, u)$ now however must be somewhat strengthened since the compact support principle deals with neighborhoods of ∞. Specifically, we shall require that, for $x \in \Omega$ and $0 \leq u < \delta$,

$$\lambda|\boldsymbol{\eta}|^2 \leq a_{ij}(x, u)\eta_i\eta_j \leq \Lambda|\boldsymbol{\eta}|^2 \tag{8.11}$$

for some positive constants λ, Λ. Moreover, for $x \in \Omega$, and for functions $u = u(x)$ such that $0 \leq u(x) < \delta$ and $|Du(x)| \leq b$ for some b, $b \geq 1$ say, we assume that

$$\|D_i a_{ij}(x, u(x))\| \leq \alpha \tag{8.12}$$

for a constant $\alpha \geq 0$.

Finally we shall suppose for the rest of the section that *any solution* u *of* (8.10) *under consideration is such that* $|Du(x)| \leq b$ *in* Ω_R *for some* $R > 0$. (This condition can be dropped if the coefficient matrix $[a_{ij}]$ is independent of u. Of course, it is to be expected that solutions $u(x)$ which approach 0 as $|x| \to \infty$ will satisfy this condition for some domain Ω_R and constant b, but this would certainly require further regularity assumptions on the equation.)

Theorem 8.5. (Compact support principle). *For the compact support principle to hold for* (8.10) *it is sufficient that* (B2) *is satisfied with* $g(u) > 0$ *for* $u > 0$*, and*

$$\int_0^\delta \frac{ds}{H^{-1}(G(s))} < \infty. \tag{8.13}$$

On the other hand, if (B1) *is satisfied with* $f(u) > 0$ *for* $u > 0$*, then for the compact support principle to hold for* (8.10) *it is necessary that* (1.7) *is satisfied.*

Proof. We first prove necessity. Here it will enough to show the existence of a radial solution $v = v(r)$ of the problem in Ω_R

$$\begin{cases} D_i\{a_{ij}(x, v(x))A(|Dv|)D_jv\} - B(x, v, Dv) \geq 0, & \text{in } \Omega_R, \\ v(R) = m, \quad v(r) \to 0 \quad \text{as } r \to \infty; & v > 0, \quad v' < 0 \quad \text{in } \Omega_R, \end{cases} \tag{8.14}$$

where (B1) holds with $f(u) > 0$ for $u > 0$ and also, by negation, condition (1.6) is satisfied.

To this end, as shown in (8.6) it is enough to consider the equation

$$[\Phi(v')]' + \frac{1}{\lambda}\left(\alpha + \kappa + \frac{n-1}{r}\Lambda\right)\Phi(v') - \frac{f(v)}{\lambda} = 0, \qquad 0 \leq v < \delta, \qquad -1 \leq v' < 0,$$

where λ and α are given by (8.11) and (8.12), respectively.

That is, the problem becomes

$$\begin{cases} [\tilde{q}(r)\Phi(v')]' - \tilde{q}(r)\tilde{f}(v) = 0, \quad \text{in } [R, \infty), \\ v(R) = m, \qquad v(r) \to 0 \quad \text{as } r \to \infty, \\ v > 0, \qquad -1 < v' < 0 \quad \text{in } \Omega_R, \end{cases} \tag{8.15}$$

where $' = d/dr$ and $\tilde{q}$, $\tilde{f}$ are given by

$$\tilde{q}(r) = r^{(n-1)\lambda^{-1}\Lambda}e^{(a+\kappa)\lambda^{-1}r}, \qquad \tilde{f}(v) = f(v)/\lambda.$$

Of course, $\tilde{f}(v)$ continues to obey (1.7), by Lemma 3.2.

The required solution can now be constructed (for suitably small m) exactly as in the proof of Theorem 5.1, with only the change that $q(r) = r^{n-1}$ is replaced by the new function $\tilde{q}(r)$, and $f(v)$ by $\tilde{f}(v)$. Note here, in particular, that

$$\frac{\tilde{q}(r)}{\tilde{q}(r+1)} = e^{-(\alpha+\kappa)/\lambda}\left(\frac{r}{r+1}\right)^{(n-1)\Lambda/\lambda},$$

which approaches the *positive* limit $e^{-(a+\kappa)\Lambda/\lambda}$ as $r \to \infty$, cf. the corresponding calculation (5.7). This completes the proof of necessity.

The proof of sufficiency is also somewhat delicate. Here the basic method is taken from Theorem 2′ of [30], with some modifications to avoid applying the superfluous technical assumption (2.5) of [30].

We first construct an appropriate radial comparison function $v = v(r)$. Fix $\sigma \in (0, 1)$ by

$$\sigma = (\Lambda + \alpha + \kappa)^{-1}.$$

We take $C < \min\{1, C_\sigma\}$ and

$$v(r) = w(R + C - r), \qquad R \leq r \leq R + C,$$

where w is the function given in Lemma 7.1, corresponding to the given values of σ and C, and of course with $f(u)$ replaces by $g(u)$. Obviously $v(R) = w(C) = \gamma\,(< \delta)$ and $v(R+C) = v'(R+C) = 0$. We can thus suppose that v is extended to all $r \geq R$ by taking $v(r) \equiv 0$ for $r > R + C$.

To check that v has the required property of an upper comparison function, we have with the help of Lemma 7.1 (and recalling that $v' \leq 0$),

$$\begin{aligned} &D_i\big\{a_{ij}(x, u(x))A(|Dv|)D_jv\big\} + \kappa\Phi(|v'|) - g(v) \\ &\qquad \leq -a_{ij}(x, u(x))\frac{x_ix_j}{r^2}[\Phi(|v'|)]' + \left(\alpha + \kappa - \Lambda\frac{n-1}{r}\right)\Phi(|v'|) - g(v) \\ &\qquad \leq a_{ij}(x, u(x))\frac{x_ix_j}{r^2}\sigma g(v) + (\alpha+\kappa)\sigma g(v) - g(v) \qquad \text{(since } C \leq 1) \\ &\qquad \leq [\sigma(\Lambda + \alpha + \kappa) - 1]g(v) = 0; \end{aligned}$$

the steps in this calculation are essentially the same as those previously used to derive (8.9).

In summary, we have

$$D_i\{a_{ij}(x,u(x))A(|Dv|)D_jv\} + \kappa\Phi(|v'|) - g(v) \le 0 \tag{8.16}$$

in Ω_R. Of course $v \equiv 0$ for $|x| \ge R_1 = R + C$, while $v > 0$ for $R \le |x| < R_1$, and $v(R) = \gamma$. It can also be observed that $\gamma = \delta$ if $C_\sigma \le 1$ but $\gamma < \delta$ otherwise.

Now consider a solution u of the inequality (8.10) in an exterior domain Ω such that $u(x) \to 0$ as $|x| \to \infty$. Under the condition (B2) it is required to show that u has compact support in Ω. We can choose $R_0 > R$ so large that $u(x) < \gamma$ in the set $\Omega_0 = \{|x| \ge R_0\}$. Then, to simplify the notation one may consider the domain Ω_0 to be the given domain Ω.

It is now enough to show that $u \le v$ as in the proof of Theorem 1.2, where v is the comparison function above, satisfying (8.16). For this purpose it is not possible to resort directly to Lemma 8.2, since $Dv \equiv 0$ for large $|x|$, while Du is unrestricted as to its null set. Accordingly we use an indirect argument.

Define $z = v - u$ in Ω. Clearly $|z| \le \gamma$. We claim that $z \ge 0$. If this is not the case, then

$$\overline{\varepsilon} = -\inf_{\Omega} z < 0, \qquad 0 < \overline{\varepsilon} \le \gamma,$$

and we shall reach a contradiction. Note first that $z = \gamma - u > 0$ when $|x| = R$, and that $z(x) \to 0$ as $|x| \to \infty$; hence the infimum of z must be attained at some (interior) point x_0 in Ω.

Define

$$\hat{\Omega} = \{R < |x| < R_1\}, \qquad \Omega_1 = \{|x| > R_1\}.$$

Then $\Omega = \hat{\Omega} \cup \partial\Omega_1 \cup \Omega_1$, so exactly the following three cases can occur:

(1) The infimum of z is attained in Ω_1.
(2) The infimum of z is not attained in Ω_1, but is reached at a point on $\partial\Omega_1$.
(3) The infimum of z is not attained in $\overline{\Omega}_1$, but is reached in $\hat{\Omega}$.

In Case 1, let the infimum be attained at x_0 in Ω_1. For x in Ω_1, define $\overline{u}(x) = -u(x) + \overline{\varepsilon}$. Then since $v \equiv 0$ in Ω_1, we see that $\overline{u} \equiv z + \overline{\varepsilon} \ge 0$ has a zero minimum at x_0. Moreover, $\overline{u}(x)$ is such that $0 \le u \equiv -\overline{u} + \overline{\varepsilon} \le \overline{\varepsilon}$, while also by (8.10)

$$D_i\{a_{ij}(x, -\overline{u} + \overline{\varepsilon})A(|D\overline{u}|)D_j\overline{u}\} + B(x, -\overline{u} + \overline{\varepsilon}, -D\overline{u}) \le 0, \qquad \overline{u} \ge 0,$$

in Ω_0. Subtracting the expression $g(-\overline{u}+\overline{\varepsilon})$ from both sides of the previous line, then gives

$$D_i\{a_{ij}(x, -\overline{u} + \overline{\varepsilon})A(|D\overline{u}|)D_j\overline{u}\} - \tilde{B}(x, \overline{u}, D\overline{u}) \le -g(-\overline{u} + \overline{\varepsilon}) \tag{8.17}$$

where

$$\tilde{B}(x, \overline{u}, \boldsymbol{\xi}) \equiv -B(x, -\overline{u} + \overline{\varepsilon}, -\boldsymbol{\xi}) + g(-\overline{u} + \overline{\varepsilon}) \le \kappa\Phi(|\boldsymbol{\xi}|),$$

using the given condition (B2) at the second step. That is, $\tilde{B}(x, \overline{u}, \boldsymbol{\xi})$ *satisfies* (B1) *with* $f \equiv 0$. Using the fact that $g(u) \ge 0$ for $0 \le u \le \gamma < \delta$, we see that $g(-\overline{u} + \overline{\varepsilon}) \ge 0$, so that finally from (8.17) there follows

$$D_i\{a_{ij}(x, -\overline{u} + \overline{\varepsilon})A(|D\overline{u}|)D_j\overline{u}\} - \tilde{B}(x, \overline{u}, D\overline{u}) \le 0$$

in Ω_0 (and hence in Ω_1). Hence by the strong maximum principle (Theorem 8.1) applied to the domain Ω_1 we obtain $\overline{u} \equiv 0$. Thus $u \equiv \overline{\varepsilon} > 0$ in Ω_1, which is impossible since $u(x) \to 0$ as $|x| \to \infty$. That is, Case 1 cannot occur.

In Case 2, let the infimum of z be reached at x_0 on $\partial\Omega_1$. In this case, obviously $\overline{u} > 0$ in Ω_1 while $\overline{u} = 0$ at x_0 (we can of course consider $\overline{u}$ as a C^1 function on $\overline{\Omega}_1$). Then, since Ω_1 clearly satisfies an interior sphere condition at x_0, the boundary lemma (Corollary 8.4) gives $\partial\overline{u}/\partial\boldsymbol{n} < 0$ at x_0. But this is also impossible, because $D\overline{u} \equiv Dz = \mathbf{0}$ at x_0.

In Case 3, necessarily $v - u = z > -\overline{\varepsilon}$ on the boundary of Ω_1, while as noted earlier $v - u > 0$ when $|x| = R_0$. Thus $v - u \ge -a$, $a \in [0, \overline{\varepsilon})$, on the boundary of $\hat{\Omega}$, while of course $u < \delta$ and $Dv \ne \mathbf{0}$ in $\hat{\Omega}$. This corresponds in essence to Lemma 8.2 for $\Omega = \hat{\Omega}$, with the roles of u and v interchanged. We can thus apply Theorem 8.1, of course for the case

$M = -a \le 0$, the conclusion being that $v - u \ge M = -a > -\bar{\varepsilon}$ in $\hat{\Omega}$. But this contradicts the condition of Case 3 that $z = v - u$ attains its infimum $-\bar{\varepsilon}$ in $\hat{\Omega}$.

We have thus shown that all three cases lead to a contradiction. Consequently $z \ge 0$ in Ω, that is $v \ge u$. In turn, $u \equiv 0$ for $|x| > R_1$, which completes the proof of the theorem. □

Corollary 8.6. *Assume that both* (B1) *and* (B2) *are satisfied and that there exists* $c > 0$ *such that* $g(u) \ge cf(u) > 0$ *for* $u > 0$. *Then the compact support principle holds for* (8.10) *if and only if* (1.7) *holds.*

We close the section with a counterexample showing the importance of the lower bound conditions (B1) and (B2). Consider the inequality

$$\Delta_p u + |Du|^{q_1} - u^{q_2} \ge 0, \qquad p > 1,\ q_1, q_2 > 0. \tag{8.18}$$

Clearly, conditions (8.13), (B1) and (B2) are satisfied if and only if $q_1 \ge p - 1$ and $q_2 < p - 1$. The compact support principle then holds for (8.18). On the other hand, for any $q_1 \in (0, p-1)$ we can take $q_1 < q_2 < p - 1$. One easily checks that (8.18) then has positive solutions $u(x) = \text{const.}\,|x|^{-l}$ on $\Omega_R = \{x \in \mathbb{R}^n \ :\ |x| > R\}$ for l and R large. Hence the compact support principle fails even though condition (1.6), or equally (8.13), is fulfilled!

9. Riemannian weighted norms

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold of class C^1, with contravariant metric tensor $[g^{ij}]$ in local coordinates $x = (x^1, \dots, x^n)$. Let u be a real–valued C^1 function defined on some open connected submanifold Ω of $\mathcal{M}$. The Riemannian norm of the gradient vector ∇u on Ω is then defined as the non–negative continuous function on Ω given in local coordinates by

$$|\nabla u|_g = \sqrt{g^{ij} D_i u D_j u}, \qquad D_i u = \frac{\partial u}{\partial x^i}.$$

Consider the variational integral

$$I[u] = \int_\Omega \{\mathcal{G}(|\nabla u|_g) + F(u)\} d\mathcal{M}.$$

The corresponding Euler–Lagrange equation is then

$$\mathrm{div}_g\{A(|\nabla u|_g)\nabla u\} - f(u) = 0, \tag{9.1}$$

where div_g is the Riemannian divergence operator and $A(\rho) = \mathcal{G}'(\rho)/\rho$, $\rho > 0$, as in the introduction, see (1.5). More explicitly, in local coordinates $x = (x^1, \dots, x^n)$ in Ω, one has $d\mathcal{M} = \sqrt{g}dx$, where $g = 1/\det[g^{ij}]$. Then a direct calculation of the Euler–Lagrange equation yields

$$\frac{1}{\sqrt{g(x)}} D_i\big(\sqrt{g(x)} g^{ij}(x) A(|\nabla u|_g) D_j u\big) - f(u) = 0, \tag{9.2}$$

that is, exactly (9.1). When $A \equiv 1$ the differential operator in (9.2) reduces just to the manifold Laplacian, see [43], page 232.

A specific example is given by the variational integral

$$\int_\Omega \left\{\frac{1}{p}|\nabla u|_g^p + F(u)\right\} d\mathcal{M}, \qquad p > 1, \qquad \text{where} \quad d\mathcal{M} = \sqrt{g}dx \text{ on } \Omega,$$

introduced by Mossino ([24], page 40), though without the volume factor $\sqrt{g}$. Here of course $A(\rho) = \rho^{p-2}$, $p > 1$. Other examples are given also in [25].

Obviously (9.2) is the special case of (1.13) when

$$a_{ij}(x, u) = \sqrt{g(x)} g^{ij}(x), \qquad B(x, u, \boldsymbol{\xi}) = \sqrt{g(x)} f(u).$$

With this motivation in hand, we turn to the strong maximum principle for (1.13). As at the beginning of the section, we assume that (A1)$'$ and (A2)$'$ are valid, and additionally that the tensors $[g^{ij}] = [g^{ij}(x, u)]$ and $[a_{ij}] = [a_{ij}(x, u)]$ are continuously differentiable, symmetric and positive definite in $\Omega \times \mathbb{R}^+_0$. In the context of (1.13) the domain Ω is now of course simply a connected open subset of $\mathbb{R}^n$.

The inequality (1.13) is more difficult to treat than (8.2), in that there are two different sets of hypotheses under which the strong maximum principle can be obtained. In the first, some mild conditions on the operator $A = A(\rho)$ are required, satisfied in particular by both the p–Laplacian operator and the mean curvature operator. In the second case, a modification of condition (B1) is needed, together with stronger conditions on the metric tensor $[g^{ij}]$. It is convenient to consider the two cases separately.

First, we introduce the additional structure hypotheses:

(A3) (i) $|A'(\rho)|\rho^2 \le c\,\Phi(\rho)$ for some constant $c \ge 0$ and for all $\rho \in (0,1]$, and
(ii) for all $\sigma_0 \in (0,1)$ there exists a value $\nu = \nu(\sigma_0)$ such that

$$\Phi'(\rho) \le \nu \Phi'(\sigma\rho)$$

for all $\sigma \in (\sigma_0, 1]$ and $\rho \in (0,1)$.

Note that if Φ is concave, condition (A3)–(ii) is always satisfied with $\nu = 1$; this is the case for example for the p–Laplacian operator when $1 < p \le 2$, and for the mean curvature operator. On the other hand, for $\Phi(\rho) = \rho^{p-1}$, $p > 2$, we get $\nu(\sigma_0) = \sigma_0^{2-p}$. It follows that (A3)–(ii) is satisfied for the p–Laplacian with $\nu = \max\{1,\ \sigma_0^{2-p}\}$.

Also (A3)–(i) is satisfied for the p–Laplacian with $c = |p-2|$ and for the mean curvature operator with $c = 1$, etc.

Theorem 9.1. *Let conditions* (A1)′, (A2)′, (A3) *and* (B1) *hold. Then the strong maximum principle is valid for inequality* (1.13) *provided that $f(s) \equiv 0$ for $s \in [0,\mu)$, $\mu > 0$, or $f(s) > 0$ for $s \in (0,\delta)$ and* (1.6) *is satisfied.*

Theorem 9.2. *Let conditions* (A1)′ *and* (A2)′ *hold, let $g^{ij} = g^{ij}(x)$ be independent of u and of class $C^2(\Omega)$, and assume* (B1) *applies with $\Phi(|\boldsymbol{\xi}|)$ replaced by $\Phi(|\boldsymbol{\xi}|_g)$. Then the strong maximum principle is valid for inequality* (1.13) *provided that $f(s) \equiv 0$ for $s \in [0,\mu)$, $\mu > 0$, or $f(s) > 0$ for $s \in (0,\delta)$ and* (1.6) *is satisfied.*

Proof of Theorem 9.1. This closely follows the proof of Theorem 8.1, though with an additional term appearing in (8.7) due to the presence of the metric $[g^{ij}]$, and with a slight (but not trivial) difference in the definition of the comparison function $v = v(r)$.

To begin with, we define the positive definite matrix $\hat{g}^{ij}(x) = g^{ij}(x, u(x))$, this of course being of class C^1 in the annular domain E_R, see the proof of Theorem 8.1. Let θ^2 and Θ^2 be respectively the least and greatest eigenvalues of the positive definite matrix $[\hat{g}^{ij}]$ in E_R, and write

$$\ell = \ell(x) = |Dr|_{\hat{g}} = \sqrt{\hat{g}^{ij}(x)x_i x_j / r^2}.$$

Then in E_R,

$$\theta \le \ell \le \Theta, \qquad (\Theta \ge 1 \text{ without loss of generality}). \tag{9.3}$$

Following the proof of Theorem 8.1, the estimate (8.5) continues to hold, and similarly, after a short calculation,

$$|\xi_k D_k \ell| \le \beta |\boldsymbol{\xi}| / \ell r \tag{9.4}$$

for some constant $\beta \ge 0$, with $\beta = 0$ if $g^{ij} = \delta_{ij}$. Finally it is convenient to define $\bar{\nu} = \nu(\theta/\Theta)$, where ν is the function given in (A3)–(ii).

Now let $v(x) = w(t)$, $t = (R-r)/\Theta$, $r = |x|$, where w is the unique solution of (4.1) given by Proposition 4.1 when $q(t) = (R - \Theta t)^k$, $T = R/2\Theta$, and f is replaced by $(\bar{\nu}\Theta^2/\lambda)f$. The constant k will be determined later.

Of course, Proposition 4.4 applies to the solution w in view of Lemma 3.2 and (1.6). Therefore $Dv(x) = -w'\boldsymbol{x}/\Theta r \ne \mathbf{0}$. Also, by restricting the boundary value $w = m$ at

$T = R/2\Theta$ to be sufficiently small, one can maintain $\|w'\|_\infty \le 1$ and so

$$0 < |Dv| \le 1 \qquad \text{in } E_R. \tag{9.5}$$

We can now turn to the important, but unfortunately somewhat complicated, calculation, applying for $x \in E_R$,

$$\begin{aligned}
D_i\{&\hat{a}_{ij}(x)A(|Dv|_{\hat{g}})D_jv\} - \kappa\Phi(|Dv|_{\hat{g}}) - f(v)\\
&= \frac{1}{\Theta^2}\hat{a}_{ij}(x)\frac{x_j}{r}\Phi'(\ell w'/\Theta)w'' \cdot \frac{x_i}{r} - A'(\ell w'/\Theta)|w'|^2 D_i\ell\\
&\qquad - \frac{1}{\Theta}D_i\left\{\hat{a}_{ij}(x)\frac{x_j}{r}\right\}A(\ell w'/\Theta)w' - \kappa\Phi(\ell w'/\Theta) - f(v)\\
&\ge \frac{1}{\Theta^2}\hat{a}_{ij}(x)\frac{x_ix_j}{r^2}\Phi'(\ell w'/\Theta)w'' - \frac{c\beta}{\ell^3 r}\Phi(\ell w'/\Theta) - \frac{\alpha R + (n-1)\Lambda}{\ell r}\Phi(\ell w'/\Theta)\\
&\qquad - \kappa\Phi(\ell w'/\Theta) - f(w) \qquad\qquad \text{(by (A3)–(i), (8.4) and (9.3))}\\
&\ge \frac{\lambda}{\bar\nu\Theta^2}\Phi'(w')w'' - \frac{1}{r}\left\{\frac{c\beta}{\theta^3} + \frac{\alpha R + (n-1)\Lambda}{\theta} + R\kappa\right\}\Phi(w') - f(w)\\
&\qquad\qquad\qquad\qquad\qquad \text{(by (9.3), (A3)–(ii) and } \Phi' > 0)\\
&= \frac{\lambda}{\bar\nu\Theta^2}\left\{[\Phi(w')]' - \frac{\bar k}{r} - \frac{\bar\nu\Theta^2}{\lambda}f(w)\right\} \qquad\qquad \text{(defining } \bar k)\\
&= \frac{\lambda}{\bar\nu\Theta^2}\left\{\frac{1}{q(t)}[q(t)\Phi(w')]' - \frac{\bar\nu\Theta^2}{\lambda}f(w)\right\} = 0,
\end{aligned}$$

where we take $k = \bar k/\Theta$.[6]

The rest of the proof is essentially the same as for Theorem 8.1, with the single exception that now the matrix $b_{kj}(\boldsymbol{\xi}) = b_{kj}(x, \boldsymbol{\xi})$ in the proof of the analogue of Lemma 8.2 is given by

$$b_{kj}(\boldsymbol{\xi}) = A(\ell|\boldsymbol{\xi}|)\delta_{ij} + \ell\frac{A'(\ell|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\xi_k\xi_j.$$

The eigenvalues of $[b_{kj}]$ are $A(\ell|\boldsymbol{\xi}|)$ and $\Phi'(\ell|\boldsymbol{\xi}|)$ so from (9.3) it is evident that $[b_{kj}]$ is positive definite for $\boldsymbol{\xi} \ne \mathbf{0}$ and all $x \in E_R$. □

Proof of Theorem 9.2. The idea of the proof is to replace the ball B_R tangent to the support of u by a small *geodesic ball* $\{x \in \Omega : s(x) \le S\}$ centered at x_0 and tangent to the singular set where $u = 0$, $Du = \mathbf{0}$; here $s(x)$ denotes the geodesic distance (with respect to the metric induced by the matrix $[g^{ij}]$) from the given center x_0 to nearby points $x \in \Omega$. The existence of such a tangent ball can be shown exactly as in Hopf's original proof, at least provided that $|Ds|$ is equally bounded above and bounded away from zero.

To show this fact, we observe by Gauss' lemma (see [43], page 235) that

$$|Ds(x)|_g^2 = g^{ij}(x)D_is(x)D_js(x) = 1, \qquad x \ne x_0. \tag{9.6}$$

Thus, recalling that θ^2 and Θ^2 are the least and greatest eigenvalues of $[g^{ij}]$, we get

$$\Theta^{-1} \le |Ds| \le \theta^{-1},$$

as required.

We can now proceed as in the proof of Theorem 9.1, with E_R replaced by the geodesic annular set $G_S = \{x \in \Omega : S/2 \le s(x) \le S\}$ and with

$$v(x) = w(t), \qquad t = S - s, \qquad T = S/2,$$

$$Dv = -w'Ds, \qquad |Dv|_g = w'$$

[6] If $g^{ij} = \delta_{ij}$, then $\ell = 1$, $\beta = 0$, $\theta = \Theta = 1$, $\bar\nu = 1$ and the calculation reduces exactly to (8.6), without the intervention of condition (A3).

by (9.6). The principal calculation, for $x \in G_S$, is the following:

$$
\begin{aligned}
&D_i\{\hat a_{ij}(x)A(|Dv|_g)D_jv\} - \kappa\Phi(|Dv|_g) - f(v)\\
&\qquad = -D_i\{\hat a_{ij}(x)D_js A(w')w'\} - \kappa\Phi(w') - f(w)\\
&\qquad = \hat a_{ij}(x)D_isD_js[\Phi(w')]' - D_i(\hat a_{ij}(x)D_is)\Phi(w') - \kappa\Phi(w') - f(w)\\
&\qquad \ge \frac{\lambda}{\Theta^2}[\Phi'(w')]' - \left(\frac{\alpha}{\theta} + \Lambda\|D^2s\|\right)\Phi(w') - \kappa\Phi(w') - f(w)\\
&\qquad \ge \frac{\lambda}{\Theta^2}[\Phi(w')]' - \frac{\bar k}{s}\Phi(w') - f(w),
\end{aligned}
\tag{9.7}
$$

where $\bar k$ is an appropriate constant. That such a constant exists depends on knowing that $s \in C^2(G_S)$, which is a consequence of the assumption that g^{ij} is of class C^2, see [43], Appendix II.1, and [33]. [Here it is essential to have g^{ij} *independent* of u, for otherwise the constructed matrix $[\hat g^{ij}]$ would be only of class C^1, however smooth the metric might be; thus in turn the corresponding geodesic distance $\hat s(x)$ would be only of class C^1 away from x_0. Of course, due to the singularity at the center x_0 the gradient Ds naturally is not continuous at x_0, while D^2s is unbounded of order $1/s$ as x approaches x_0 (always assuming that g^{ij} is of class C^2). These comments are reflected in the trivial $\mathbb{R}^n$ calculation that $Dr = \boldsymbol{x}/r$ is not continuous at the singularity $x_0 = 0$, though it is bounded, and that the Hessian matrix $D^2r = r^{-1}[\delta_{ij} - x_ix_j/r^2]_{ij}$.

The existence of the constant $\bar k$ being shown, one can choose $w = w(t)$ so that the right side of (9.7) vanishes, and the rest of the proof follows as before. The fact that $\Phi(|\boldsymbol{\xi}|_g)$ replaces $\Phi(|\boldsymbol{\xi}|)$ in condition (B1) causes no difficulty in the application of Theorems 8.1 and 10.1, since for $|\boldsymbol{\xi}| \le |\boldsymbol{\eta}|$ there results

$$\Phi(|\boldsymbol{\eta}|_g) - \Phi(|\boldsymbol{\xi}|_g) \le (\Theta^2/\theta)\Phi'(\Theta|\boldsymbol{\eta}|)\cdot|\boldsymbol{\eta} - \boldsymbol{\xi}|,$$

that is $\Phi(|\boldsymbol{\xi}|_g)$, as well as $\Phi(|\boldsymbol{\xi}|)$, is Lipschitz continuous in $\boldsymbol{\xi}$. □

The strong maximum principle for the Riemannian equation (9.1), or for the corresponding inequality

$$\operatorname{div}_g\{A(|\nabla u|_g)\nabla u\} - f(u) \le 0 \qquad \text{in } \Omega, \tag{9.8}$$

can be treated more simply than for the case of inequality (1.13), and under slightly lighter hypotheses. The result is as follows.

Theorem 9.3. *Let conditions* (A1), (A2) *and* (F2) *hold. Assume that the Riemannian manifold $\mathcal{M}$ is of class C^2. Then the strong maximum principle is valid for inequality* (9.8) *provided that $f(s) \equiv 0$ for $s \in [0,\mu)$, $\mu > 0$, or $f(s) > 0$ for $s \in (0,\delta)$ and* (1.6) *is satisfied.*

Proof. We begin as in the proof of Theorem 9.2, with the exception that (9.7) now becomes more simply, for $x \in G_S$,

$$
\begin{aligned}
&\frac{1}{\sqrt{g(x)}}D_i\{\sqrt{g(x)}g^{ij}(x)A(|Dv|_g)D_jv\} - f(v)\\
&\qquad = -\frac{1}{\sqrt{g(x)}}D_i\{\sqrt{g(x)}g^{ij}(x)D_js\, A(w')w'\} - f(w)\\
&\qquad = [\Phi(w')]' - \Delta s\,\Phi(w') - f(w) \ge [\Phi(w')]' - \frac{\bar k}{s}\Phi(w') - f(w).
\end{aligned}
\tag{9.9}
$$

The remaining part of the proof involves the weak comparison theorem. In the present case this can be done with the help of Theorem 10.5 rather than the more difficult Theorem 10.1. To this end, we have to check (10.10) when $\hat{\boldsymbol{A}}(x,\boldsymbol{\xi}) = \sqrt{g(x)}g^{ij}(x)A(|\boldsymbol{\xi}|_g)\boldsymbol{\xi}$, that is, in Riemannian notation,

$$\sqrt{g(x)}\langle A(|\boldsymbol{\eta}|_g)\boldsymbol{\eta} - A(|\boldsymbol{\xi}|_g)\boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\xi}\rangle_{\mathcal{M}} \ge \sqrt{g(x)}\big(\Phi(|\boldsymbol{\eta}|_g) - \Phi(|\boldsymbol{\xi}|_g)\big)\cdot\big(|\boldsymbol{\eta}|_g - |\boldsymbol{\xi}|_g\big)$$

since $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle_{\mathcal{M}} \le |\boldsymbol{\xi}|_g |\boldsymbol{\eta}|_g$, and (10.10) now follows because Φ is strictly increasing by (A2). $\square$

In [25] a version of the strong maximum principle at infinity, the so–called Omori–Yau principle, has recently been given for singular elliptic inequalities including the p–Laplacian case as well as the mean curvature operator, and for smooth, connected, non–compact, complete Riemannian manifolds $\mathcal{M}$.

10. Comparison and uniqueness theorems for singular divergence form operators

10.1. **Comparison results**

Throughout the section we consider the pair of differential inequalities

$$\operatorname{div}\{\hat{\boldsymbol{A}}(x, u, Du)\} - \hat{B}(x, u, Du) \le 0, \qquad u \ge 0, \tag{10.1}$$

$$\operatorname{div}\{\hat{\boldsymbol{A}}(x, v, Dv)\} - \hat{B}(x, v, Dv) \ge 0, \qquad v \ge 0, \tag{10.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. Let the vector function

$$\hat{\boldsymbol{A}}(x, z, \boldsymbol{\xi}) : \ \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

be continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and continuously differentiable with respect to z and $\boldsymbol{\xi}$ for all z and for $\boldsymbol{\xi} \ne \boldsymbol{0}$. Also let

$$\hat{B}(x, z, \boldsymbol{\xi}) : \ \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

be continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and continuously differentiable with respect to $\boldsymbol{\xi}$ for $|\boldsymbol{\xi}| > 0$ in $\mathbb{R}^n$. Suppose moreover throughout the section that $\hat{\boldsymbol{A}}$ is elliptic in the sense that the matrix $[D_{\xi_j} \hat{A}^i(x, z, \boldsymbol{\xi})]$ is positive definite for $x \in \Omega$ and $\boldsymbol{\xi} \ne \boldsymbol{0}$ in $\mathbb{R}^n$. Finally *assume that $\hat{B}(x, z, \boldsymbol{\xi})$ is non–decreasing in the variable z for $x \in \Omega$ and $|\boldsymbol{\xi}| \le b$.*

Then the following comparison principle holds.

Theorem 10.1. (Comparison principle). *Let u and v be respective solutions of* (10.1) *and* (10.2) *in Ω. Suppose that u and v are continuous in $\overline{\Omega}$, with $|Du| + |Dv| > 0$ in Ω, and either $|Du| < b$ or $|Dv| < b$. Assume finally $u \ge v$ on $\partial\Omega$.*

If $\hat{\boldsymbol{A}}$ is independent of the variable z, then $u \ge v$ in Ω.

More generally if the boundary condition is relaxed to $u \ge v - M$ on $\partial\Omega$, where M is constant, then $u \ge v - M$ in Ω.

This is essentially Theorem 10.7 (i) of [18] with the exception that the functions $\hat{\boldsymbol{A}}$ and $\hat{B}$ are allowed to be singular at $\boldsymbol{\xi} = \boldsymbol{0}$, this being compensated by the additional condition $|Du| + |Dv| > 0$ in Ω. We have written $\hat{\boldsymbol{A}}$, $\hat{B}$ here, rather than $\boldsymbol{A}$, B as in [18], in order to avoid confusion with earlier notation in the paper.

If Ω is unbounded, the boundary condition is understood to include the limit relation

$$\liminf \ \{u(x) - v(x)\} \ge -M \qquad \text{as } |x| \to \infty.$$

Before giving the proof it is convenient to state the following

Lemma 10.2. *Let $\hat{\Omega}$ be a compact subset of Ω, and $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ vectors in $\mathbb{R}^n$ satisfying*

$$|\boldsymbol{\xi}|, \, |\boldsymbol{\eta}| \le b, \qquad |t\boldsymbol{\xi} + (1-t)\boldsymbol{\eta}| \ge d$$

for some positive constants b and d, with $d \le b$, and for all $t \in (0, 1)$. Also suppose $|z| \le \ell$. Then there exist constants ν, ν^* depending only on b, d, ℓ and $\hat{\Omega}$ such that

$$\{\hat{\boldsymbol{A}}(x, z, \boldsymbol{\xi}) - \hat{\boldsymbol{A}}(x, z, \boldsymbol{\eta})\} \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \ge \nu |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \tag{10.3}$$

and

$$|\hat{B}(x, z, \boldsymbol{\xi}) - \hat{B}(x, z, \boldsymbol{\eta})| \le \nu^* |\boldsymbol{\xi} - \boldsymbol{\eta}|. \tag{10.4}$$

Proof. By the integral mean value theorem,

$$\hat{\boldsymbol{A}}(x,z,\boldsymbol{\xi}) - \hat{\boldsymbol{A}}(x,z,\boldsymbol{\eta}) = \int_0^1 D_{\xi_j}\hat{\boldsymbol{A}}(x,z,\,t\boldsymbol{\xi} + (1-t)\boldsymbol{\eta})(\xi_j - \eta_j)dt.$$

But the matrix $[D_j\hat{A}^i(x,z,\boldsymbol{\zeta})]$ is *uniformly* positive definite for x in $\hat{\Omega}$, $|z| \le \ell$ and $d \le |\boldsymbol{\zeta}| \le b$, and the first conclusion then follows at once.

Similarly

$$\hat{B}(x,z,\boldsymbol{\xi}) - \hat{B}(x,z,\boldsymbol{\eta}) = \int_0^1 D_{\xi_j}\hat{B}(x,z,\,t\boldsymbol{\xi} + (1-t)\boldsymbol{\eta})(\xi_j - \eta_j)dt.$$

Here $D_{\xi_j}\hat{B}(x,z,\boldsymbol{\zeta})$ is *uniformly* bounded for x in $\hat{\Omega}$, $|z| \le \ell$ and $d \le |\boldsymbol{\zeta}| \le b$, and the second inequality is proved. □

It may be remarked that in the special case of the p–Laplacian operator, that is, when $\hat{\boldsymbol{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi}$, we can take $\nu = d^{p-2}$ when $p \ge 2$, and $\nu^* = (p-1)b^{p-2}$ when $p < 2$.

Proof of Theorem 10.1. It is enough to treat $M = 0$, since the case for arbitrary values of M reduces to $M = 0$ by the substitution $\overline{v} = v - M$.

Now suppose for contradiction that the conclusion is false. Put $w(x) = u(x) - v(x)$, whence

$$\overline{\varepsilon} = -\inf_{x\in\Omega} w(x) > 0.$$

Then for $\varepsilon \in (\overline{\varepsilon}/2,\, \overline{\varepsilon})$ the function

$$w_\varepsilon = \min\{w + \varepsilon,\, 0\}$$

is non–vanishing exactly in the set

$$\Sigma = \Sigma_\varepsilon = \{x \in \Omega :\ w_\varepsilon(x) < 0\}.$$

Since $w + \varepsilon > 0$ on $\partial\Omega$ it is evident that Σ is pre–compact in Ω.

We assert that if ε is suitably close to $\overline{\varepsilon}$ then

$$|tDu + (1-t)Dv| \ge d, \qquad |Du|,\, |Dv| \le b, \tag{10.5}$$

in Σ, where $d > 0$ is a constant (independent of ε) such that $|Du| + |Dv| \ge 4d$ in the pre–compact set $\Sigma_{\overline{a}}$. To see this, observe first that $Du - Dv = Dw = \mathbf{0}$ on the closed subset $E = \{x \in \Omega :\ w(x) = -\overline{\varepsilon}\}$ of Σ. Moreover, distance$(E, \partial\Sigma) \to 0$ as $\varepsilon \to \overline{\varepsilon}$. Hence by continuity, $|Du - Dv| < d$ in Σ provided ε ($> \overline{\varepsilon}/2$) is suitably near $\overline{\varepsilon}$. In particular, for such values of ε we find (since surely $\max\{|Du|, |Dv|\} \ge 2d$ in Σ)

$$|tDu + (1-t)Dv| \ge \max\{|Du|, |Dv|\} - |Du - Dv| \ge d \quad \text{in } \Sigma,$$

which is the first part of (10.5).

For the second part, consider (without loss of generality) the case where $|Dv| < b$ in Ω. Define $\overline{b} = \sup_{x\in\Sigma_{\overline{\varepsilon}/2}} |Dv(x)|$. Then $\overline{b} < b$, and if we choose ε even nearer to $\overline{\varepsilon}$, if necessary, then also $|Du - Dv| < b - \overline{b}$ in Σ. But then $|Du| \le |Dv| + |Du - Dv| \le b$ in Σ, as required.

Continuing now as in the proof of Theorem 5.4, and using the non–positive test function w_ε, we have

$$\begin{aligned} \int_\Omega \{\hat{\boldsymbol{A}}(x,Du) - \hat{\boldsymbol{A}}(x,Dv)\}Dw_\varepsilon &\le \int_\Sigma \{\hat{B}(x,v,Dv) - \hat{B}(x,u,Du)\}w_\varepsilon \\ &\le \int_\Sigma \{\hat{B}(x,u,Dv) - \hat{B}(x,u,Du)\}w_\varepsilon, \end{aligned} \tag{10.6}$$

where in the last step of (10.6) we have used the facts that $w_\varepsilon \le 0$ and $u \le v$ in Σ, and that $\hat{B}$ is non–decreasing in the variable z. Then, with the help of Lemma 10.2, (10.6) implies that

$$\nu \int_\Sigma |Dw_\varepsilon|^2 \le \nu^* \int_\Sigma |Dw_\varepsilon| \cdot |w_\varepsilon|. \tag{10.7}$$

Let $\Gamma = \Gamma_\varepsilon = \{\varepsilon - \overline{\varepsilon} < w_\varepsilon < 0\}$. Then $Dw_\varepsilon = \mathbf{0}$ on $\Sigma \setminus \Gamma = E$, so the integrals in (10.7) can equally be taken over the set Γ.

Applying the Cauchy–Schwarz inequality to the right side of (10.7) yields

$$(\nu^*/\nu)^2 \int_\Gamma |w_\varepsilon|^2 \geq \int_\Gamma |Dw_\varepsilon|^2. \tag{10.8}$$

From Poincaré's inequality (cf. (7.44) on page 164 of [18]) we obtain

$$\omega_n^{-1}|\Gamma|^{1/n}\,||Dw_\varepsilon||_{\Gamma,2} = \omega_n^{-1}|\Gamma|^{1/n}\,||Dw_\varepsilon||_{\Sigma,2} \geq ||w_\varepsilon||_{\Sigma,2} \geq ||w_\varepsilon||_{\Gamma,2}.$$

Hence by (10.8) there results

$$|\Gamma| \geq \omega_n(\nu/\nu^*)^n. \tag{10.9}$$

On the other hand, $\Gamma \to \emptyset$ as $\varepsilon \to \overline{\varepsilon}$, a contradiction to (10.9). This completes the proof. □

In the following two theorems, the stated conditions on Du and Dv in Theorem 8.1 are removed. Essentially similar results were given earlier by Damascelli [9]; see also [10].

Theorem 10.3. (Comparison principle). *Suppose that $\hat{\boldsymbol{A}}$ is independent of u, and that the matrix $[\partial \hat{A}^i/\partial \xi_j]$ is uniformly positive definite when $0 < |\boldsymbol{\xi}| \leq$ Const., u is bounded and x is in any compact subset of Ω. Assume additionally that $\hat{B}$ is uniformly Lipschitz continuous with respect to $\boldsymbol{\xi}$ on compact subsets of its variables and is non–decreasing in the variable u.*

If $u \geq v - M$ on $\partial\Omega$, where M is constant, then $u \geq v - M$ in Ω.

To prove Theorem 10.3 it is enough to observe that the conclusions of Lemma 10.2 hold without the restriction $|t\boldsymbol{\xi} + (1-t)\boldsymbol{\eta}| \geq d$. In fact if $\boldsymbol{\xi} = \boldsymbol{\eta} = \mathbf{0}$ then (10.3) and (10.4) are trivially true, while otherwise certainly $|t\boldsymbol{\xi} + (1-t)\boldsymbol{\eta}| > 0$, in which case the conclusions follows from the hypothesis of uniformly positive definiteness and the Lipschitz continuity of $\hat{B}$.

This being shown, the proof of Theorem 10.1 then carries over unchanged, without the intervention of (10.5).

The special case of the p–Laplacian operator is of particular importance. This is given in the following

Corollary 10.4. *Consider the inequalities*

$$\begin{aligned} \Delta_p u - \hat{B}(x, u, Du) &\leq 0 \qquad \text{in } \Omega, \\ \Delta_p v - \hat{B}(x, v, Dv) &\geq 0 \qquad \text{in } \Omega, \end{aligned}$$

where $p \leq 2$, and $\hat{B} = \hat{B}(x, z, \boldsymbol{\xi})$ is uniformly Lipschitz continuous in $\boldsymbol{\xi}$ on compact subsets of its variables (and of course non–decreasing in the variable z). If $u \geq v - M$ on $\partial\Omega$, where M is constant, then $u \geq v - M$ in Ω.

Remark. If in Theorems 10.1 and 10.3 one adds the hypothesis that $u \geq 0$, $v < \delta$, then the monotonicity of $\hat{B}$ is needed only in the interval $0 \leq z < \delta - M$; see the proof of Theorem 5.4.

Theorem 10.5. (Comparison principle). *Let u and v be respective solutions of* (10.1) *and* (10.2) *in Ω. Suppose that u and v are continuous in $\overline{\Omega}$, that $\hat{\boldsymbol{A}}$ is independent of z and $\hat{B}$ is independent of $\boldsymbol{\xi}$. Assume moreover that $\hat{\boldsymbol{A}}$ is monotone in the variable $\boldsymbol{\xi}$ (but not necessarily differentiable), i.e.*

$$\{\hat{\boldsymbol{A}}(x, \boldsymbol{\xi}) - \hat{\boldsymbol{A}}(x, \boldsymbol{\eta})\} \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) > 0, \qquad \text{when } \boldsymbol{\xi} \neq \boldsymbol{\eta}. \tag{10.10}$$

If $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.

This follows at once from (10.10), exactly as in the proof of Theorem 5.4.

Strong comparison theorems, under alternative hypotheses, have been obtained by Tolksdorf [38] and by Cuesta and Takáč [8].

There is a final comparison theorem which avoids the conditions on Du and Dv in Theorem 10.1, but at the expense of a simpler boundary condition.

Theorem 10.6. *Let u be a solution of the inequality*

$$D_i\{a_{ij}(x,u)A(|Du|)D_ju\} - B(x,u,Du) \le 0 \qquad \text{in } \Omega.$$

Suppose that (8.1) *is satisfied and that*

$$B(x,z,\boldsymbol{\xi}) \le \kappa\Phi(|\boldsymbol{\xi}|) \tag{10.11}$$

for $x \in \Omega$, $z < 0$, and $|\boldsymbol{\xi}| \le 1$.

If $u \ge 0$ on $\partial\Omega$ then $u \ge 0$ in Ω.

Proof. Assume for contradiction that u has a negative minimum M at some point x_0 in Ω. Put $w = u - M$. Then $w \ge 0$ in Ω, while $w(x_0) = 0$. Using (10.11) one sees that w is a solution of the inequality

$$D_i\{a_{ij}(x,w+M)A(|Dw|)D_jw\} - \kappa\Phi(|Dw|) \le 0$$

in some neighborhood N of x_0 (where $0 \le w < |M|$). Hence by Theorem 8.1 we find $w \equiv 0$ in N, and then by chaining also $w \equiv 0$ in Ω, which is impossible by the boundary condition. □

Theorem 10.6 is false without condition (10.11), as follows from the example, equation (1.10), in the introduction. Indeed, essentially as noted there, this equation has the solution $u(x) = C(|x|^k - 1)$ on the unit ball, which vanishes on the boundary, and at the same time is negative in the interior.

While we have not found a proof, we conjecture that the full result of Theorem 10.1 should hold without the stated conditions on Du and Dv provided that $\hat{B}$ obeys (B1).

10.2. **Uniqueness of the Dirichlet problem**

The structure built up in the earlier parts of this section, and also in previous sections, allows one to present a number of uniqueness theorems for the Dirichlet problem

$$\begin{aligned} &\operatorname{div}\hat{\boldsymbol{A}}(x,u,Du) - \hat{B}(x,u,Du) = 0 \qquad && \text{in } \Omega,\\ &u(x) = \varphi(x) && \text{on } \partial\Omega, \end{aligned} \tag{10.12}$$

where $\varphi \in C(\partial\Omega)$.

Theorem 10.7. *Suppose $\hat{\boldsymbol{A}}$ is independent of u and $\hat{B}$ of Du, and that* (10.10) *holds. Then problem* (10.12) *can have at most one solution.*

This is an immediate consequence of Theorem 10.5. The special case when $\hat{\boldsymbol{A}}(x,u,\boldsymbol{\xi}) = A(|\boldsymbol{\xi}|)\boldsymbol{\xi}$ (and A satisfies conditions (A1) and (A2) in the introduction, for example the case of the p–Laplacian) also follows directly from Theorem 5.4.

Theorem 10.8. *Suppose that $\hat{\boldsymbol{A}}$ is independent of u, and that the matrix $[\partial\hat{A}_i/\partial\xi_j]$ is uniformly positive definite when $0 < |\boldsymbol{\xi}| \le$ Const. and x is in any compact subset of Ω. Assume additionally that $\hat{B}$ is uniformly Lipschitz continuous with respect to $\boldsymbol{\xi}$ on compact subsets of its variables (and of course non–decreasing in the variable u).*

Then the problem (10.12) *can have at most one solution.*

The special case of the p–Laplacian operator is of particular importance. This is given in the following

Corollary 10.9. *The Dirichlet problem*

$$\begin{aligned} \Delta_p u - \hat{B}(x,u,Du) &= 0 \qquad && \text{in } \Omega, \\ u(x) &= \varphi(x) && \text{on } \partial\Omega, \end{aligned}$$

where $p \le 2$ and $\hat{B} = \hat{B}(x,z,\boldsymbol{\xi})$ is uniformly Lipschitz continuous in $\boldsymbol{\xi}$ on compact subsets of its variables, can have at most one solution.

When the boundary data takes the canonical form $u = 0$ on $\partial\Omega$, then the condition of uniform positive definiteness in the previous theorem can be dropped. The result is as follows.

Theorem 10.10. *Consider the equation*

$$D_i\{a_{ij}(x,u)A(|Du|)D_j u\} - B(x,u,Du) = 0 \qquad \text{in } \Omega,$$

with (8.1) *satisfied. Assume also*

$$[\operatorname{sign} z] \cdot B(x,z,\boldsymbol{\xi}) \ge -\kappa\Phi(|\boldsymbol{\xi}|) \tag{10.13}$$

for $x \in \Omega$, $z \in \mathbb{R}$ and $|\boldsymbol{\xi}| \le 1$. Then the Dirichlet problem $u = 0$ on $\partial\Omega$ has the unique solution $u \equiv 0$.

This follows immediately from Theorem 10.6, once it is shown that $u \equiv 0$ is a solution. But this is a consequence of the fact that $B(x,0,\mathbf{0}) = 0$. Indeed by (10.13) one has

$$[\operatorname{sign} z] \cdot B(x,z,\mathbf{0}) \ge 0$$

so that $B(x,z,\mathbf{0})$ changes sign as z passes through zero, which by continuity gives $B(x,0,\mathbf{0}) = 0$.

11. p–REGULAR EQUATIONS

For a large set of equations displaying p–homogeneity, $p > 1$, including in particular equations involving the p–Laplacian Δ_p, there is an elegant Strong Maximum Principle which corresponds closely to the case of regular equations discussed in the introduction.

In particular, we consider the singular differential inequality

$$\operatorname{div}\hat{\boldsymbol{A}}(x,u,Du) - \hat{B}(x,u,Du) \le 0 \quad \text{in } \Omega, \qquad u \ge 0, \tag{11.1}$$

where the (measurable) functions $\hat{\boldsymbol{A}}$ and $\hat{B}$ have the following homogeneity and ellipticity properties for all $x \in \mathcal{D}$, $u \in \mathbb{R}_0^+$ and $\boldsymbol{\xi} \in \mathbb{R}^n$

$$\begin{aligned} \hat{\boldsymbol{A}}(x,u,\boldsymbol{\xi}) \cdot \boldsymbol{\xi} &\ge a_1|\boldsymbol{\xi}|^p - a_2 u^p \\ |\hat{\boldsymbol{A}}(x,u,\boldsymbol{\xi})| &\le a_3|\boldsymbol{\xi}|^{p-1} + a_4 u^{p-1} \\ \hat{B}(x,u,\boldsymbol{\xi}) &\le b_1|\boldsymbol{\xi}|^{p-1} + b_2 u^{p-1} \end{aligned} \tag{11.2}$$

with a_1, $a_3 > 0$; a_2, a_4, b_1, $b_2 \ge 0$ (see [35], where these conditions apparently appear first).

Trudinger [39], closely using the ideas of [35], has proved under these conditions the following beautiful Harnack inequality for continuous (non–negative) solutions u of (11.1) which are in the Sobolev space $W^{1,p}(\Omega)$:

For any ball B_R, such that $0 < R \le 1$ and $B_{3R} \subset \Omega$, there holds

$$||u||_{B_{2R},\gamma} \le C|R|^{n/\gamma} \min_{B_R} u(x), \tag{11.3}$$

where C depends only $(p,n,\gamma,a_1,a_2,a_3,a_4,b_1,b_2)$ and $\gamma \in (0,(p-1)n/(n-p))$ (or $(0,\infty)$ if $p \ge n$).

This immediately implies the following Strong Maximum Principle.[7]

Theorem 11.1. (Strong maximum principle). *Let u be a (non–negative) solution of* (11.1) *in Ω, as defined above. Then either $u \equiv 0$ in Ω or $u > 0$ in Ω.*

[7] The special case $a_2 = a_4 = 0$ and $\hat{B} = 0$ was noted by Granlund [19].

Proof. Indeed suppose that $u = 0$ at some point x_0 in Ω. Let B_{3R} be a ball centered at x_0, with R so small that B_{3R} is in Ω. Then $\min_{B_R} u(x) = 0$, so in turn $||u||_{B_{2R},\gamma} = 0$ by (11.3).

That is, $u = 0$ in B_{2R}. Chaining then gives the conclusion $u \equiv 0$ in Ω, proving the theorem. □

Remark. If we consider *classical distribution solutions* of (11.1), rather than the weaker class above, then conditions (11.2) need only apply for *small* $u \geq 0$, say $u < \delta$, and for $|\boldsymbol{\xi}| \leq 1$, say.

To prove Theorem 11.1 for this case, we first modify $\hat{\boldsymbol{A}}$ and $\hat{B}$ for values $u \geq \delta$ and $|\boldsymbol{\xi}| > 1$, so that the modified functions remain measurable but now also satisfy (11.2) for the complete set of variables. Then, corresponding to any classical (non–negative) solution of (11.1) for which $u(x_0) = 0$, there is some neighborhood N of x_0 where $u < \delta$ and $|\boldsymbol{\xi}| \leq 1$. Therefore u satisfies the modified equation in N, for which the full conditions (11.2) hold. Thus $u \equiv 0$ in N by Theorem 11.1, and then $u \equiv 0$ in Ω, by chaining.

Theorem 11.1 is obviously broad and powerful. On the other hand, it has some drawbacks in comparison with Theorems 1.1 and 1.2 (or Theorem 8.1 and 8.5). Specifically it applies only to operators $A(\rho)$ which obey

$$\text{Const.}\,\rho^{p-1} \leq \Phi(\rho) \leq \text{Const.}\,\rho^{p-1} \tag{11.4}$$

for some positive constants, and similarly it requires that the function $f(u)$ in (1.1), or in (B1), must satisfy $f(u) \leq u^{p-1}$ for small $u > 0$. Finally, of course, it does not lend itself to the precise necessary and sufficient condition (1.6), even in case A obeys (11.4)

There is a corresponding comparison theorem of interest, valid under the stronger conditions following:

$$\hat{\boldsymbol{A}}(x, u, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq a_1|\boldsymbol{\xi}|^p, \qquad \hat{B}(x, u, \boldsymbol{\xi}) \leq b_1|\boldsymbol{\xi}|^{p-1}, \tag{11.5}$$

where $a_1 > 0$ and $b_1 \geq 0$.

Theorem 11.2. (Comparison principle). *Let u be a solution of the inequality* (11.1), *where $\hat{\boldsymbol{A}}$ and $\hat{B}$ satisfy* (11.5) *for $x \in \Omega$ and $u < M$.*

If $u \geq M$ on $\partial\Omega$, then $u \geq M$ in Ω.

Gilbarg and Trudinger give a related result ([18], Theorem 10.9), but with a more difficult proof.

Proof. Suppose for contradiction that the result fails. We then follow the proof of Theorem 10.1, with $w(x) = u(x) - M$, however without the intervention of (10.5). Corresponding to (10.6), one finds, using the non–positive test function w_ε,

$$\int_\Omega \hat{\boldsymbol{A}}(x, u, Du) \cdot Dw_\varepsilon \leq -\int_\Sigma \hat{B}(x, u, Du) w_\varepsilon. \tag{11.6}$$

Then, with the help of (11.5) and the fact that $Du = Dw_\epsilon$ on Σ, the inequality (11.6) implies that

$$a_1 \int_\Sigma |Dw_\varepsilon|^p \leq b_1 \int_\Sigma |Dw_\varepsilon|^{p-1} \cdot |w_\varepsilon|. \tag{11.7}$$

Let $\Gamma = \Gamma_\varepsilon = \{\varepsilon - \bar{\varepsilon} < w_\varepsilon < 0\}$. Then $Dw_\varepsilon = \mathbf{0}$ on $\Sigma \setminus \Gamma = E$, so the integrals in (10.7) can equally be taken over the set Γ.

Applying Hőlder's inequality to the right side of (11.7) yields, cf. (10.8),

$$b_1||w_\varepsilon||_{\Gamma,p} \geq a_1||Dw_\varepsilon||_{\Gamma,p}. \tag{11.8}$$

From Poincaré's inequality (7.44) of [18], we obtain

$$\omega_n^{-1}|\Gamma|^{1/n}\,||Dw_\varepsilon||_{\Gamma,p} = \omega_n^{-1}|\Gamma|^{1/n}\,||Dw_\varepsilon||_{\Sigma,p} \geq ||w_\varepsilon||_{\Sigma,2p} \geq ||w_\varepsilon||_{\Gamma,2p}.$$

Hence by (11.8) there results

$$|\Gamma| \geq \omega_n(a_1/b_1)^n. \tag{11.9}$$

On the other hand, $\Gamma \to \emptyset$ as $\varepsilon \to \overline{\varepsilon}$, a contradiction to (11.9). This completes the proof. $\square$

12. Special cases

12.1. The linear case

Consider the linear inequality

$$D_i\{a_{ij}(x)D_ju\} + b_i(x)D_iu + c(x)u \le 0, \qquad u \ge 0, \tag{12.1}$$

for $x \in \Omega$, where the matrix $[a_{ij}]$ is continuously differentiable and satisfies (8.1), b_i, $c \in C(\Omega)$ for all $i = 1, \dots, n$. This is the special case of (8.2) where $A(\rho) \equiv 1$, $B(x, u, \boldsymbol{\xi}) = -b_i(x)\xi_i - c(x)u$. Here we can apply the result of Theorem 8.1, assuming also that $b_i(x)$ and $c(x)$ are locally bounded. By slightly shrinking the domain Ω we can then suppose that

$$\kappa = \max_i \sup_\Omega |b_i(x)| < \infty, \qquad c = -\inf_\Omega \{c(x),\ 0\} < \infty,$$

and moreover define $f(u) = cu$. Then $\Phi(\rho) = \rho$, $H^{-1}(\rho) = \sqrt{2\rho}$ and $F(u) = cu^2/2$, so that (B1) and (1.6) hold as required. This gives the strong maximum principle for (12.1), closely related to the classical Theorem 2.2 of E. Hopf. Indeed, assuming as above that a_{ij} is continuously differentiable, then the strong maximum principle for C^2 solutions of (12.1) is an immediate consequence of Theorem 2.2, while conversely the strong maximum principle for C^1 distribution solutions of (2.1) follows at once from Theorem 8.1.

These comments moreover lead us to expect that the proof of Theorem 8.1 can be simplified for the special linear case. In fact, the principal inequality (8.6) in the proof of Theorem 8.1 suggests that the required comparison function v for the Hopf proof can be obtained for the linear case by exhibiting an explicit solution of the inequality

$$\{|v'|\}' + \frac{k}{r}|v'| + \frac{cv}{\lambda} \le 0.$$

(since $\Phi(\rho) = \rho$ in the present linear case). A natural choice for v is

$$v(r) = \alpha\left[\left(\frac{R}{r}\right)^{\vartheta} - 1\right], \qquad \frac{R}{2} \le r \le R, \tag{12.2}$$

where ϑ and R are to be determined. Then $v'(r) = \dfrac{\alpha\vartheta}{R}\left(\dfrac{R}{r}\right)^{\vartheta+1}$ and so after a short calculation

$$\begin{aligned} |v'|' + \frac{k}{r}|v'| + \frac{cv}{\lambda} &= \alpha\vartheta\left(\frac{R}{r}\right)^{\vartheta}\left\{\frac{k-(\vartheta+1)}{r^2}\right\} + \frac{cv}{\lambda} \\ &\le \alpha\vartheta\left(\frac{R}{r}\right)^{\vartheta}\left\{\frac{k-(\vartheta+1)}{r^2} + \frac{c}{\lambda\vartheta}\right\}. \end{aligned}$$

This will be ≤ 0 provided that

$$\vartheta = 2k - 1, \qquad R^2 \le \frac{\lambda k(2k-1)}{c}.$$

Thus the rational comparison function (12.2) can be used for the linear inequality (12.1), alternative to the standard exponential function

$$v(r) = \varepsilon\big(e^{-\alpha r^2} - e^{-\alpha R^2}\big),$$

see page 148 of [20], or page 34 of [18].

12.2. **The degenerate Laplacian case**

A similar simplification can be used for the canonical inequality

$$\Delta_p u - f(u) \le 0, \qquad u \ge 0, \tag{12.3}$$

for the p–Laplace operator, $p > 1$. For our present purpose, we assume that

$$f(u) \le cu^{p-1}, \tag{12.4}$$

the borderline case for (1.6).

The comparison function $v = v(r)$, $r = |x|$, for (6.1) again can be taken in the form (12.2). Then we have

$$\Phi(|v'|) = |v'|^{p-1} = \left(\frac{\alpha\vartheta}{R}\right)^{p-1}\left(\frac{R}{r}\right)^{(p-1)(\vartheta+1)};$$

Thus as before, we find after a short calculation that

$$[\Phi(|v'|)]' + \frac{n-1}{r}\Phi(|v'|) + \frac{f(v)}{\lambda} \le (\alpha\vartheta)^{p-1}\left(\frac{R}{r}\right)^{(p-1)\vartheta}\left\{\frac{n-1-(p-1)(\vartheta+1)}{r^p} + \frac{c}{\lambda\vartheta^{p-1}}\right\}.$$

This again will be ≤ 0 provided that

$$\vartheta = \frac{2(n-1)}{p-1} - 1, \qquad R \le \left(\frac{(n-1)\lambda}{c}\right)^{1/p}\vartheta^{1/p'}.$$

That is, $\Delta_p v - f(v) \ge 0$ for $R/2 \le |x| \le R$, and the proof of the strong maximum principle, Theorem 1.1, now applies unchanged, but without using Proposition 4.1.

In summary, for the borderline case (12.4) of inequality (12.3), we get an elementary proof of Vázquez' strong maximum principle, avoiding the delicate arguments of Sections 3 and 4, or of [41].

Note that the simple comparison function (12.2) does not suffice for general operators or for more complicated nonlinearities. This observation indicates the need for the new construction of $v = v(r)$ used in the proof of Theorem 1.1. Of course, for more complicated linearities it is also necessary to use the comparison Theorem 10.1 rather than the simpler Theorem 5.4.

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