Some New Results on Global Nonexistence for Abstract Evolution Equations with Positive Initial Energy

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Dedicated to Olga Ladyzhenskaya with admiration and esteem

§1. Introduction.

In a recent paper [7] the problem of non–continuation was studied for abstract evolution equations of the type

$$P u_{tt} + Q(t) u_t + A(t,u) = F(t,u), \quad t \in J = [0,\infty), \tag{1.1}$$

where P and $Q(t)$ are linear self-adjoint operators, and $A(t,u)$ and $F(t,u)$ are typically a divergence operator in u and a nonlinear driving force.

Other versions of (1.1) were considered earlier by Levine [3–6], for which he introduced the important technique of "concavity" analysis of auxiliary second order differential inequalities. In all these papers the principal mechanism of blow–up was the assumption of negative initial energy.

In an interesting paper [10], which has just appeared, Ono has also used concavity analysis to study blow–up, but in the more general case when the initial energy is allowed to take appropriately small positive values. His analysis primarily considers linear wave operators, and moreover is restricted to bounded domains in $\mathbb{R}^n$. (It should, however, be added that Ono also allows Kirchhoff type operators, an added generalization but without serious affect on the principal ideas.)

Here we discuss some extentions of Ono's analysis to the abstract equation (1.1), see Theorem 1. Moreover, in concrete cases, we introduce appropriate methods to treat divergence structure operators in unbounded domains (including but not necessarily restricted to $\mathbb{R}^n$). Our conclusions also yield a larger class of initial data than in [10] for which blow-up must occur; see Remark 1 in Section 3.

In the next section we give a precise meaning to equation (1.1), and give our main abstract theorem. Section 3 discusses a divergence structure equation in $\mathbb{R}^n$ for which blow–up occurs for positive initial energy, even for unbounded domains. Here the primary new idea, in comparison with [7] and [10], is to introduce an appropriate coercive operator associated with the equation.

Proofs of the results described here will appear in the forthcoming paper [13].

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[1] This work is partially supported by CNR, Progetto Startegico *Modelli e Metodi per la Matematica e l'Ingegneria* and by the Italian *Ministero della Università e della Ricerca Scientifica e Tecnologica* under the auspices of *Gruppo Nazionale di Analisi Funzionale e sue Applicazioni* of the CNR.

§2. The main theorem.

Let X be a Banach space, and X' its dual space. If $x \in X$ and $x' \in X'$, we shall write $\langle x', x\rangle_X$ to denote the natural pairing of x and x', that is $\langle x', x\rangle_X = x'(x)$.

Let V be a Hilbert space. An operator $P : V \to V'$ will be called *symmetric* if

$$\langle Pv, w\rangle_V = \langle Pw, v\rangle_V \quad \text{for all } v, w \in V,$$

and *non–negative definite* if

$$\langle Pv, v\rangle_V \geq 0 \quad \text{for all } v \in V.$$

It is easy to check that a symmetric operator must be linear and, moreover, continuous by the uniform boundedness theorem.

We consider the evolution equation (1.1), where P is symmetric and non–negative definite from V into V'. We suppose that the *dissipation operator* $Q(t)$ is, for each $t \in J$, symmetric and non–negative definite from an appropriate Hilbert space Y into its dual Y'. In addition, assume $Q \in C(J \to B(Y, Y'))$, that is $\langle Q(\cdot)v, w\rangle_Y : J \to \mathbb{R}$ is continuous for each $v, w \in Y$. Note that $P \equiv 0$ and $Q \equiv 0$ are specifically allowed.

Finally, the operators A and F are such that[2]

$$A : J \times W \to W', \qquad F : J \times X \to X',$$

with W, X Banach spaces and W', X' their duals. In order to define the energy $\mathcal{E}u$ of a solution of (1.1), see below, it is necessary that there exist C^1 potentials

$$\mathcal{A} : J \times W \to \mathbb{R}, \qquad \mathcal{F} : J \times X \to \mathbb{R},$$

such that for each fixed t the operators A and F are the Fréchet derivatives with respect to u of $\mathcal{A}$ and $\mathcal{F}$, respectively; by normalization we can take $\mathcal{A}(t, 0) \equiv 0$, $\mathcal{F}(t, 0) \equiv 0$.

Now suppose that there is given a nontrivial subspace G of V, W, X and Y – not necessarily closed. Let

$$K = \{\varphi : J \to G \mid \varphi \in C(J \to W) \cap C(J \to X) \cap C^1(J \to V) \cap AC(J \to Y)\}.$$

We say that u is a (*strong*) *solution* of (1.1) if

(a) $u \in K$;

(b) Distribution Identity:

$$\langle Pu_t(\tau), \varphi(\tau)\rangle_V \Big|_0^t = \int_0^t \{\langle Pu_t(\tau), \varphi_t(\tau)\rangle_V - \langle Q(\tau)u_t(\tau), \varphi(\tau)\rangle_Y$$
$$- \langle A(\tau, u(\tau)), \varphi(\tau)\rangle_W + \langle F(\tau, u(\tau)), \varphi(\tau)\rangle_X\} d\tau$$

[2]Specific examples are given in [4, 7, 8, 11], and also in Section 3 below. For further clarity and definiteness we refer the reader to these papers.

for all $t \in J$ and $\varphi \in K$;
(c) Energy Conservation:

$$\mathcal{E}u(t) - \mathcal{E}u(0) \leq -\int_0^t \{\langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y - \mathcal{A}_t(\tau, u(\tau)) + \mathcal{F}_t(\tau, u(\tau))\} d\tau, \tag{2.1}$$

where

$$\mathcal{E}u(t) = \tfrac{1}{2}\langle Pu_t(t), u_t(t)\rangle_V + \mathcal{A}(t, u(t)) - \mathcal{F}(t, u(t)), \quad t \in J, \tag{2.2}$$

is the total energy of u.

Assume even more that $Q \in C^1(J \to B(Y, Y'))$, with $Q_t(t) : Y \to Y'$ being non–positive definite and (necessarily) symmetric for all $t \in J$.

Theorem 1. *Suppose there are constants $p \geq q$ such that, for all $(t, u) \in J \times G$,*

$$\langle A(t, u), u\rangle_W - \langle F(t, u), u\rangle_X \leq q\mathcal{A}(t, u) - p\mathcal{F}(t, u) \tag{2.3}$$

and

$$\mathcal{A}_t(t, u) - \mathcal{F}_t(t, u) \leq 0. \tag{2.4}$$

Let $p > 2$. Then there is no solution $u = u(t)$ of (1.1) *on J with*

$$\mathcal{A}(t, u(t)) \geq \lambda_0 > 0, \qquad t \in J, \tag{2.5}$$

and

$$\mathcal{E}u(0) < \left(1 - \frac{q}{p}\right)\lambda_0 = D_0.$$

Remarks. **1.** If $p = q$ in (2.3), then Theorem 1 remains valid with $D_0 = 0$, but without requiring (2.5). In other words, non–continuation holds under the single condition $\mathcal{E}u(0) < 0$, namely, negative initial energy; this is exactly the main result of [7].

2. In the usual applications $\mathcal{A}$ is independent of t, in which case (2.4) reduces simply to $\mathcal{F}_t(t, u) \geq 0$ on $J \times G$.

§3. Examples.

Now let Ω be an open domain in $\mathbb{R}^n$ and consider the model problem

$$u_{tt} - \operatorname{div}(|Du|^{q-2}Du) + \mu|u|^{q-2}u = f(t, x, u), \quad x \in \Omega, \ t \in J, \tag{3.1}$$

where

$$f(t, x, u) = g(t, x)|u|^{\sigma-2}u + c|u|^{p-2}u, \tag{3.2}$$

and

$$\mu \geq 0, \quad 1 < q < p; \qquad c > 0, \quad 1 < \sigma < p. \tag{3.3}$$

For the function g we assume

$$-g,\ \frac{\partial g}{\partial t} \geq 0 \quad \text{on } J \times \Omega, \qquad g(t,\cdot) \in L^{p/(p-\sigma)}(\Omega) \quad \text{for all } t \in J. \tag{3.4}$$

Here the appropriate spaces are $V = L^2(\Omega)$, $W = W_0^{1,q}(\Omega)$, $X = L^p(\Omega)$ and $G = L^2(\Omega) \cap L^p(\Omega) \cap W_0^{1,q}(\Omega)$. For definiteness the space W will be endowed with the norm

$$\|u\|_W = (\|u\|_{L^q(\Omega)}^q + \|Du\|_{L^q(\Omega)}^q)^{1/q}.$$

(Note that the remaining space Y is unneeded, since for simplicity we have omitted damping terms from the equation. In fact, adding a term $a(t,x)u_t$, $a \geq 0$, $\partial a/\partial t \leq 0$ on $J \times \Omega$, to the left hand side of (3.1) leaves Theorems 2 and 3 below unchanged.)

The operator P corresponding to (3.1) is given by $\langle Pv, w\rangle_V = (v,w)_{L^2}$; clearly P is symmetric and positive definite. We take $A(u) = -\text{div}(|Du|^{q-2}Du) + \mu|u|^{q-2}u$, so that[3]

$$\langle A(u), u\rangle_W = \|Du\|_{L^q}^q + \mu\|u\|_{L^q}^q = q\mathcal{A}(u).$$

On the other hand, as is easy to see, one must then have

$$\mathcal{F}(t,u) = \frac{1}{\sigma}\int_\Omega g(t,x)|u|^\sigma dx + \frac{c}{p}\|u\|_{L^p}^p. \tag{3.5}$$

By a solution of (3.1) we now mean a (strong) solution of the abstract evolution equation (1.1) corresponding to the operators P, A and F just defined.

Clearly (2.4) is satisfied on $J \times G$ by $(3.4)_1$ – we assume suitable regularity of $g = g(t,x)$ so that $\mathcal{F}_t(t,u)$ can be calculated on $J \times G$ by differentiation under the integral sign in (3.5). Finally,

$$\langle F(t,u), u\rangle_X = \int_\Omega g(t,x)|u|^\sigma dx + c\|u\|_{L^p}^p,$$

so $p\mathcal{F}(t,u) \geq \langle F(t,u), u\rangle_X$ because $\sigma < p$ and $g \leq 0$. Therefore (2.3) is verified on $J \times G$.

Theorem 2. *Assume the conditions* (3.3), (3.4), *and the further restrictions* $\mu > 0$, $2 < p \leq r$ *hold, where* $r = nq/(n-q)$ *is the Sobolev exponent for* $W_0^{1,q}(\Omega)$ *when* $q < n$,

[3]The choice $A(u) = -\text{div}(|Du|^{q-2}Du)$ is inconvenient when $\Omega = \mathbb{R}^n$, or indeed when Ω has infinite measure, due to the lack of coercivity.

or otherwise $q < p < \infty$ *if* $q \geq n$. *Then the problem* (3.1), (3.2) *cannot have any global solution* u *corresponding to initial data*

$$\mathcal{E}u(0) < E_0, \qquad \|u(0)\|_{L^p} > \Lambda_0, \tag{3.6}$$

where E_0, Λ_0 *are given by*

$$\Lambda_0^{p-q} = \frac{qB}{c}, \qquad E_0 = B\Lambda_0^q\left(1 - \frac{q}{p}\right),$$

and B *is the best constant for the coercive estimate*

$$\mathcal{A}(u) \geq B\|u\|_{L^p}^q, \qquad u \in W_0^{1,q}(\Omega). \tag{3.7}$$

The existence of $B > 0$ for which (3.7) holds is a consequence of the Sobolev embedding theorem (Adams [1, Theorem 5.4, Part III, pp. 97–98]). The corresponding result for bounded domains Ω is slightly different.

Theorem 3. *Let the measure of* Ω *be finite. Then Theorem 2 remains valid even when* $\mu = 0$.

Remarks. 1. Ono in [10] essentially treats the semilinear case $q = 2$, $\mu = 0$ of Theorem 3, but with the initial data satisfying the somewhat stronger conditions

$$\mathcal{E}u(0) < E_0, \qquad \|Du(0)\|_{L^2}^2 < c\|u(0)\|_{L^p}^p.$$

Other work involving positive initial energies appears earlier in [2] and [9], the first however restricted to the wave operator itself (with nonlinear boundary conditions), and the second with less precise bounds for the initial energy.

2. The results above make clear the distinction between the case when $\mathcal{E}u(0)$ is taken to be negative, and when it is allowed to be positive, as in (3.6). In particular, in the latter case it is necessary that the potential $\mathcal{A}(u)$ be coercive so that in turn one must assume that $p \leq r$, a condition which was not needed in the corresponding examples in Section 4 of [7].

3. In the example concerning the degenerate s–Laplacian on page 262 of [7] the condition $s > 2$ was required for the application of their Theorem 1. Here we assume only that $s = q > 1$. This significant improvement is made possible by the more general form of condition (2.3) here in comparison with the corresponding assumption of [7].

4. The results of Theorems 2 and 3 arise directly from the algebraic behavior of the function $E(\lambda)$ representing the potential well for (3.1). For initial data which lies deep enough in the "well" itself, the corresponding solutions are asymptotically stable. This dichotomy is discussed in detail in [12].

A number of concrete examples relative to linear operators A were given in Section III of [4], to which we refer the reader. Example VI of [4, p.16] in particular deserves special mention. Here the operator $-Q$ is the Laplacian, so for precision the space Y as well as W must be chosen as $H_0^1(\Omega)$.

Other concrete operators $A(u)$ are given in Section 6 of [11], notably the polyharmonic operator $(-\Delta)^L$, where $L \geq 1$ is an integer, and still further examples are given in Section 4 of [8].

All of these examples allow extensions to the time dependent case and to positive initial energies, for both bounded and unbounded domains, as discussed above.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press (1975).

[2] R.J. Knops, H.A. Levine & L.E. Payne, *Non–existence, instability and growth theorems for solutions of a class of abstract nonlinear equations with applications to nonlinear elastodynamics*, Archive Rational Mech. Anal. **55** (1974), 52–72.

[3] H.A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = Au + \mathcal{F}(u)$, Archive Rational Mech. Anal. **51** (1973), 371–386.

[4] H.A. Levine, *Instability and nonexistence of global solutions of nonlinear wave equations of the form* $Pu_{tt} = Au + \mathcal{F}(u)$, Trans. Amer. Math. Soc. **192** (1974), 1–21.

[5] H.A. Levine, *Some additional remarks on the nonexistence of global solutions to nonlinear wave equations*, SIAM J. Math. Anal. **5** (1974), 138–146.

[6] H.A. Levine, *Nonexistence of global weak solutions to nonlinear wave equations*, in *Improperly Posed Boundary Value Problems*, Res. Notes Math. **1**, 94–104; Pitman, London, 1975.

[7] H.A. Levine, P. Pucci & J. Serrin, *Some remarks on global nonexistence for nonautonomous abstract evolution equations*, Contemporary Math. **208** (1997), 253–263.

[8] H.A. Levine & J. Serrin, *Global nonexistence theorems for quasilinear evolution equations with dissipation*, Archive Rational Mech. Anal. **137** (1997), 341–361.

[9] H.A. Levine & R.A. Smith, *A potential well theory for the wave equation with a nonlinear boundary condition*, J. Reine Angew. Math. **374** (1987), 1–23.

[10] K. Ono, *Global existence, decay, and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings*, J. Diff. Eqs. **137** (1997), 273–301.

[11] P. Pucci & J. Serrin, *Asymptotic stability for non–autonomous dissipative wave systems*, Comm. Pure Appl. Math. **XLIX** (1996), 177–216.

[12] P. Pucci & J. Serrin, *Local asymptotic stability for dissipative wave systems*, Israel J. Math. (1998) 1–22, in press.

[13] P. Pucci & J. Serrin, *Global nonexistence for abstract evolution equations with positive initial energy*, J. Diff. Equations, to appear.

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