The Harnack inequality in $\mathbb{R}^2$ *for quasilinear elliptic equations*

P. Pucci & J. Serrin

Abstract. We use maximum principle techniques to obtain a Harnack inequality for two dimensional elliptic operators.

1. Introduction

We use maximum principle techniques to obtain a Harnack inequality. Although the final result is restricted to plane domains, on the other hand the class of equations to which it applies is in many respects more extensive than in previous work. We consider the model quasilinear elliptic differential equation

$$\text{div}\{A(|Du|)Du\} - f(u) = 0, \qquad u > 0, \tag{1.1}$$

in $\mathbb{R}^2$, and more generally the two sided differential inequality

$$-B(Du) \leq \text{div}\{A(|Du|)Du\} \leq B(Du) + f(u), \qquad u > 0. \tag{1.2}$$

Throughout the paper we let $\Omega(t) = tA(t)$ for $t > 0$ and $\Omega(0) = 0$, and assume the natural conditions

(Ω1) $\Omega \in C[0, \infty) \cap C^{1,1}(0, \infty)$.
(Ω2) $\Omega'(t) > 0$ for $t > 0$ (Ellipticity).
(B) B is continuous in $\mathbb{R}^2$ and continuously differentiable in $\mathbb{R}^2 \setminus \{0\}$. Moreover

$$0 \leq B(p) \leq \mu\Omega(|p|) \quad (\mu = \text{Pos. Const.}) \qquad \text{for all } p \in \mathbb{R}^2 \text{ with } |p| < 1.$$

(F) f is continuous and non–decreasing on $[0, \infty)$, with $f(0) = 0$.

In addition to these assumptions, several further technical conditions will be placed on the function Ω to allow the derivations to proceed. It is convenient to defer their statements for the moment (see (Ω3) – (Ω4) at the end of the introduction). We note particularly, however, that (1.1) is a special case of (1.2), and also that the prime examples of the $m-$ *Laplace operator* $A(t) = t^{m-2}$, $m > 1$, and the *mean curvature operator* $A(t) = 1/\sqrt{1+t^2}$ are covered by our hypotheses, see below.

Now define

$$F(u) = \int_0^u f(v)dv, \qquad u \geq 0,$$

and

$$G(t) = \int_0^t \Omega(s)ds, \qquad H(t) = tG'(t) - G(t) = \int_0^t s\Omega'(s)ds, \quad t \geq 0.$$

Clearly F is non-negative, while G is strictly convex and H is strictly increasing and positive by (Ω2). Let us denote by B_R the disk in $\mathbb{R}^2$ centered at $(0,0)$ with radius $R > 0$. Then the following main result holds.

Theorem 1. *Let the function Ω satisfy* (Ω1)–(Ω4)*, let B satisfy condition* (B) *and assume f obeys* (F)*. Also suppose either $f \equiv 0$ on $[0, \delta]$, $\delta > 0$, or that $f > 0$ on $\mathbb{R}^+$ and*

$$\int_0^1 \frac{ds}{H^{-1}(F(s))} = \infty. \tag{1.3}$$

Then for every $R > 0$ there is a strictly increasing continuous function $\Phi_R : [0, \infty) \to [0, \infty)$, with $\Phi_R(0) = 0$, such that any C^1 distribution solution u of the differential inequality (1.2) *in the disk B_R satisfies the Harnack condition*

$$u(x,y) \geq \Phi_R(u(0,0)) \qquad \text{for all } (x,y) \in B_{R/3}. \tag{1.4}$$

Our method of proof does not make obvious the dependence of the function Φ_R on R or on the parameters of (1.2). For a given function A, however, it is worth noting that the dependence of Φ_R on f arises from the divergence rate as $t \to 0$ of the integral

$$\int_t^1 \frac{ds}{H^{-1}(F(s))}.$$

The main previous results concerning the Harnack inequality for general quasilinear elliptic operators are due to the second author of this paper [S2] and to Lieberman [L], both of whom studied the general divergence structure equation

$$\text{div} A(x, u, Du) + B(x, u, Du) = 0 \tag{1.5}$$

in n–dimensional space. In comparison with that work, the restriction here to $\mathbb{R}^2$ is a serious drawback. On the other hand, for the inequality (1.2) itself, conditions (Ω1)–(Ω4) are weaker than those earlier required; for example, in [S2] the operator $A(t)$ is closely related to the degenerate Laplace case $\Omega(t) = t^{m-1}$, while on the other hand Lieberman asks that (Ω1)–(Ω3) and [L, (1.1)$'$] hold. In our notation, (1.1)$'$ takes the form

$$t\Omega'(t) \geq \delta\Omega(t), \qquad t \geq 0, \qquad \delta > 0, \tag{1.6}$$

and in particular (Ω2)–(Ω3) and (1.6) imply (Ω4) with $c_1(\theta) = c\theta^{c-1}/\delta$.

It should be noted as well that the mean curvature operator does not obey the conditions of [L], see [L, (1.1)$'$]. More importantly, in [L] the Harnack principle depends on a given bound $0 \leq u \leq M$ for the solution, a strong condition which is not required here. Finally, our basic requirement on the nonlinearity f, namely the integral condition (1.3), is more general than the assumptions of either [S2] or [L]. [Suppose, e.g., that f satisfies Lieberman's conditions (1.1)$'$ (that is, (1.6) above) and (1.3c)$''$. Then by (1.3c)$''$ we have

$$0 \le f(u) \le \gamma\Omega(u), \qquad u \ge 0,$$

so with the help of (1.6) one easily derives

$$H(t) \ge \frac{\delta}{\delta+1} t\Omega(t), \quad t \ge 0; \qquad F(u) \le \gamma u\Omega(u), \quad u \ge 0.$$

Hence $F(u) \le H(du)$ for $d = \max\{1, (\delta+1)\gamma/\delta\}$. Thus $H^{-1}(F(u)) \le du$ and so (1.3) holds.]

For the special case of the mean curvature and other closely related operators, Trudinger [T] has also given a Harnack inequality in n dimensions, but again however only for bounded solutions, with the constant in the Harnack principle depending on the bound. Moreover, in the context of inequality (1.2) and the mean curvature operator $A(t) = 1/\sqrt{1+t^2}$, Trudinger requires for example the conditions $0 \le B(p) \le \text{Const.}\{|p|^2/(1+|p|) + |p|/(1+|p|)^{1/2}\}$ and $0 \le f(u) \le \text{Const.}\, u$, see [T, (3.2)], which are respectively stronger than conditions (B) and (1.3). Finally, we note reference [G], which considers viscosity solutions of the minimal surface and other related equations, always when $f \equiv 0$.

Theorem 1 has the following important companion result.

Theorem 2. *Under the assumptions of Theorem* 1*, for every $R > 0$ there exists a strictly increasing continuous function $\Psi_R : [0, M_R) \to [0, \infty)$, with $\Psi_R(0) = 0$, such that any C^1 distribution solution u of* (1.2) *in the disk B_R, with $u(0,0) < M_R$, satisfies*

$$u(x,y) \le \Psi_R(u(0,0)) \qquad \textit{for all } (x,y) \in B_{R/4}. \tag{1.7}$$

In Proposition 3 and Theorem 3 (Sections 2 and 3) we give a criterion for the value M_R to be infinite, in which case the restriction $u(0,0) < M_R$ in Theorem 2 can be omitted.

The importance of condition (1.3) for the Harnack inequality is emphasized by its appearence also in the Keller–Osserman theorem [K], [O] (case of the Laplace operator) and in the strong maximum principle for quasilinear equations [V], [D], [PSZ]. It should be noted as well that the standard extension of the Harnack inequality to arbitrary domains is also true in the present circumstances.

The further conditions for the function Ω, whose statement has previously been deferred, can now be given.

(Ω3) There is a positive constant c such that

$$t\Omega'(t) \le c\Omega(t) \quad \text{for all } t > 0;$$

(Ω4) For all $\theta \geq 1$ there is a number $c_1 = c_1(\theta) \geq 1$ such that

$$\Omega'(\lambda t) \leq c_1 \Omega'(t) \quad \text{for all } \lambda \in [1,\, \theta] \text{ and } t > 0.$$

Conditions (Ω1) – (Ω4) are notably weak. For example, they hold for the following examples:

1. The generalized mean curvature operator $A(t) = (1+t^2)^{-s/2}$, $s \leq 1$. Here one can take $c = \max\{1,\ 1-s\}$; similarly $c_1 = 1$ if $s = 1$ and $c_1 = \theta^q$, $q = \max\{2,\ -s\}$ if $s < 1$. Note also that $\Omega(\infty) = 1$ for $s = 1$, while $\Omega(\infty) = \infty$ for $s < 1$.

2. The m–Laplace operator $A(t) = t^{m-2}$, $m > 1$. Here one takes $c = m - 1$, $c_1 = \theta^{m-2}$.

3. $A(t) = t^{m-2} + t^{m_1-2}$, $1 < m \leq m_1$. Now $c = m_1 - 1$ and $c_1 = \theta^{m_1-2}$. See [BFP] for applications in quantum physics.

4. $A(t) = \tan^{-1} t/t$. Here $c = 1$ and $c_1 = 1$, while $\Omega(\infty) = \pi/2$.

5. $A(t) = (1+t^2)^{-s/2} t^{m-2}$, $s \leq m-1$, $m > 1$. Then $c = \max\{m-1,\ m-s-1\}$ and $c_1 = \theta^q$, $q = \max\{m,\ m-s-2\}$. Also $\Omega(\infty) = 1$ for $s = m-1$, otherwise $\Omega(\infty) = \infty$.

In the final section of the paper we apply our results to the Euler equation for the variational problem

$$\delta \int a(x,y,u)\big[G(|Du|) + F(u)\big]\, dxdy = 0,$$

where the integrands $G(t)$ and $F(u)$ are as given above and $a = a(x,y,u)$ is a positive C^1 function of its variables.

2. Construction of comparison functions

The proof of Theorem 1 uses the method given in [S1]. Let E denote the closed region bounded by an arbitrary ellipse in $\mathbb{R}^2$. It is convenient to introduce *local Cartesian coordinates* (ξ, η) on E, so that E is given by

$$\sigma^2 = \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} \leq 1, \qquad a \geq b > 0$$

(note that the coordinates (ξ, η) on E are related by translation and rotation to the global coordinates (x, y) on $\mathbb{R}^2$). The subset

$$\tfrac{1}{2} \leq \sigma \leq 1, \qquad \xi \geq \tfrac{1}{2} a$$

of E will be called a *minorant region* and denoted by Σ. To each minorant region Σ we shall associate a C^2 function $v = v(\sigma) : \Sigma \to \mathbb{R}^+$ (where $\sigma \in [\frac{1}{2},\, 1]$ is the local coordinate on Σ defined above), called a *minorant function* and constructed in such a way that

$$\begin{aligned} &\operatorname{div}\{A(|Dv|Dv)\} - B(Dv) - f(v) \geq 0, \quad |Dv| < 1 \qquad \text{in } \Sigma, \\ &v(1) = 0; \qquad v' < 0, \quad v'' \geq 0 \quad \text{in } [\tfrac{1}{2},\, 1]; \end{aligned} \tag{2.1}$$

here $'$ denotes differentiation with respect to the variable σ.[1]

To carry out the construction, we shall need the following lemmas; it is convenient in stating and proving these to replace the variables (ξ, η) by (x, y) - the corresponding results for the variables (ξ, η) then follow from the translational and rotational invariance of (2.1).

Lemma 1. *Let $\sigma^2 = x^2/a^2 + y^2/b^2$, $s^2 = x^2/a^4 + y^2/b^4$. Then if $v = v(\sigma)$ and $v'(\sigma) < 0$, we have*

$$(i) \qquad Dv = \left(\frac{x}{a^2}, \frac{y}{b^2}\right)\frac{v'}{\sigma}, \qquad |Dv| = \frac{s}{\sigma}|v'|,$$

$$(ii) \qquad A(|Dv|)Dv = -\frac{1}{s}\left(\frac{x}{a^2}, \frac{y}{b^2}\right)\Omega(|Dv|),$$

$$(iii) \qquad \operatorname{div}\left\{\frac{1}{s}\left(\frac{x}{a^2}, \frac{y}{b^2}\right)\right\} = \frac{1}{a^2b^2}\frac{\sigma^2}{s^3},$$

$$(iv) \qquad \frac{1}{s}\left(\frac{x}{a^2}, \frac{y}{b^2}\right)\cdot D\{\Omega(|Dv|)\} = \left(\frac{a^2-b^2}{a^3b^3}\right)^2\frac{x^2y^2}{\sigma^3 s^2}\,\Omega'(|Dv|)|v'| - \frac{s^2}{\sigma^2}\,\Omega'(|Dv|)v''.$$

Proof. The formulas in (i) are obvious, and (ii) then follows at once since $v' < 0$ by assumption. Identity (iii) is proved by direct calculation. Again a straightforward calculation and the use of (i) gives

$$\frac{\sigma}{s}\left(\frac{x}{a^2}, \frac{y}{b^2}\right)\cdot D\{\Omega(|Dv|)\} = -\left\{\left[\frac{1}{s^2}\left(\frac{x^2}{a^6} + \frac{y^2}{b^6}\right) - \frac{s^2}{\sigma^2}\right]v' + \frac{s^2}{\sigma}v''\right\}\Omega'(|Dv|).$$

But

$$\sigma^2\left(\frac{x^2}{a^6} + \frac{y^2}{b^6}\right) - s^4 = \left(\frac{a^2-b^2}{a^3b^3}\right)^2 x^2y^2,$$

which immediately gives (iv), since $v' < 0$.

Lemma 2. *We have*

$$(i) \quad bs \le \sigma \le as, \qquad\qquad (ii) \quad \left(\frac{xy}{s\sigma}\right)^2 \le \frac{a^3b^3}{4}.$$

Proof. Condition (i) follows at once from the definition of s and σ, and the assumption $a \ge b > 0$. To prove (ii) note that

$$\frac{1}{a^3b^3}\left(\frac{xy}{s\sigma}\right)^2 = \frac{1}{\sigma^2}\left(\frac{x}{a}\frac{y}{b}\right)\cdot\frac{1}{s^2}\left(\frac{x}{a^2}\frac{y}{b^2}\right) \le \frac{1}{4},$$

[1]By standard abuse of notation, the letter v denotes either the mapping $v : \Sigma \to \mathbb{R}^+$ or the ordinary real function $v : [\frac{1}{2}, 1] \to \mathbb{R}^+$. The distinction will always be clear form the context.

which follows by applying the Cauchy–Schwarz inequality to the two products in parentheses.

Lemma 3. *Let* ($\Omega 3$) *and* ($\Omega 4$) *hold. Then*

$$(i) \qquad \Omega'(|Dv|)|v'| \le ac\Omega\left(|v'|/b\right),$$

$$(ii) \qquad \frac{s^2}{\sigma^2}\Omega'(|Dv|) \ge c_2\Omega'\left(|v'|/b\right), \qquad \text{where } c_2^{-1} = a^2 c_1(a/b).$$

Proof. To prove (i) we observe by Lemma 1 (i) that

$$\begin{aligned}\Omega'(|Dv|)|v'| = \Omega'((s/\sigma)|v'|)|v'| &= \frac{\sigma}{s}\Omega'(t)t \qquad \big(t = (s/\sigma)|v'|\big)\\ &\le ac\Omega(t) = ac\Omega((s/\sigma)|v'|) \le ac\Omega(|v'|/b)\end{aligned}$$

by Lemma 2 (i) and assumptions ($\Omega 2$), ($\Omega 3$), proving (i). To obtain (ii) note that

$$\frac{s^2}{\sigma^2}\Omega'(|Dv|) = \frac{s^2}{\sigma^2}\Omega'((s/\sigma)|v'|) \ge \frac{1}{a^2}\Omega'((s/\sigma)|v'|) \ge c_2\Omega'(|v'|/b),$$

where at the second step we have used Lemma 2 (i), and at the final step ($\Omega 4$) with $t = (s/\sigma)|v'|$ and $\lambda = \sigma/bs \in [1, a/b]$.

Lemma 4. *Let* $w = v/b$. *Then*

$$-\text{div}\{A(|Dv|)Dv\} \le \frac{1}{\ell}\{\Omega(|w'|)\}' + \frac{\kappa}{\ell\sigma}\Omega(|w'|),$$

where

$$\kappa = c_1(\theta)\left[\theta^3 + \tfrac{1}{4}\left(\theta^2 - 1\right)^2 c\right], \qquad \ell = a\theta c_1(\theta), \qquad \theta = a/b.$$

Proof. By direct calculation, using Lemma 1 $(ii) - (iv)$,

$$\begin{aligned}-\text{div}\{A(|Dv|)Dv\} &= \Omega(|Dv|)\text{div}\left\{\frac{1}{s}\left(\frac{x}{a^2}, \frac{y}{b^2}\right)\right\} + \frac{1}{s}\left(\frac{x}{a^2}, \frac{y}{b^2}\right)\cdot D\{\Omega(|Dv|)\}\\ &= \frac{1}{a^2b^2}\frac{\sigma^2}{s^3}\Omega(|Dv|) + \left(\frac{a^2 - b^2}{a^3b^3}\right)^2\frac{x^2y^2}{\sigma^3 s^2}\,\Omega'(|Dv|)|v'| - \frac{s^2}{\sigma^2}\,\Omega'(|Dv|)v''\\ &\le \frac{1}{a^2b^2}\frac{a^3}{\sigma}\Omega(|Dv|) + \frac{1}{a^3b^3}\left(\frac{a^2 - b^2}{2}\right)^2\frac{1}{\sigma}\Omega'(|Dv|)|v'| - \frac{s^2}{\sigma^2}\,\Omega'(|Dv|)v''\end{aligned}$$

by Lemma 2 (i), (ii). In turn, from Lemma 2 (i) again and Lemma 3 (i), (ii) we get

$$\begin{aligned}-\text{div}\{A(|Dv|)Dv\} &\le \frac{1}{a\theta\sigma}\left[\theta^3 + \tfrac{1}{4}(\theta^2 - 1)^2 c\right]\Omega(|v'|/b) - c_2\Omega'(|v'|/b)v''\\ &= \frac{\kappa}{\ell\sigma}\Omega(|v'|/b) + \frac{1}{\ell}\{\Omega(|v'|/b)\}',\end{aligned}$$

where κ, ℓ and θ are as given in the statement of Lemma 4, and where we have used the facts that $v' < 0$, $v'' \geq 0$ and $\Omega'(t) > 0$ for $t > 0$. Lemma 4 follows since $w = v/b$.

We can now complete the construction of the minorant function $v = bw$ by choosing w as a solution of the following initial value problem (see (2.1) and Lemma 4):

$$\{\Omega(|w'|)\}' + \frac{k}{\sigma}\Omega(|w'|) + \ell f(bw) = 0, \qquad k = \kappa + \mu\ell, \tag{2.2}$$

$$w(1) = 0, \qquad w'(1) = -\alpha < 0. \tag{2.3}$$

A short calculation then shows that if w is a solution of the problem (2.2)–(2.3) over the *full interval* $[\frac{1}{2}, 1]$, and if $w' > -1$, then $v = bw : \Sigma \to \mathbb{R}^+$ is a minorant function on Σ. (Here one uses condition (B), the relation $|Dv| = (s/\sigma)|v'| = (bs/\sigma)|w'| \leq |w'| < 1$ and $\Omega(|Dv|) \leq \Omega(|w'|)$, since Ω is increasing.)

The content of the next result is that such a solution w exists.

Proposition 1. *Suppose either $f \equiv 0$ on $[0, \delta]$, $\delta > 0$, or that $f > 0$ on $\mathbb{R}^+$ and* (1.3) *holds.*

Then there exists a number $\alpha_0 > 0$ such that the initial value problem (2.2) – (2.3) *has a unique convex C^2 solution $w = w_\alpha = w_\alpha(\sigma)$ on the entire interval $[\frac{1}{2}, 1]$, with $w' > -1$, provided that $\alpha \in (0, \alpha_0)$.*

Moreover w_α depends continuously on α;

$$w'_\alpha < 0, \quad w''_\alpha \geq 0 \quad \text{in } [\tfrac{1}{2}, 1];$$

$$w_\alpha > w_\beta \quad \text{if } \alpha > \beta > 0;$$

$$w_\alpha(\sigma) \geq \alpha(1-\sigma), \qquad \lim_{\alpha \to 0} w_\alpha(\tfrac{1}{2}) = 0;$$

and the limit $m_0 = \lim_{\alpha \to \alpha_0} w_\alpha(\frac{1}{2})$ exists (finite).

Proof. 1. *Existence.* By Lemma 2 of [PS] there exists a convex piecewise C^2 solution $z = z(\sigma)$, $\sigma \in [\frac{1}{2}, 1]$, of the differential inequality

$$\{\Omega(|z'|)\}' + \frac{k}{\sigma}\Omega(|z'|) + \ell f(bz) \leq 0; \qquad z(1) = 0, \quad z'(1) = -\beta, \tag{2.4}$$

provided that β is sufficiently small, say $\beta < \beta_0$ for some appropriate positive number β_0. From [PS] it also follows that z, $z' \to 0$ uniformly as $\beta \to 0$.

We can now construct the required solution w of (2.2)–(2.3) by a standard continuation procedure. Fix $\alpha > 0$. By the Schauder fixed point argument, see [FLS, Proposition A1], there exists a C^2 solution w of (2.2), (2.3) on some non-trivial interval $(\sigma_0, 1]$. Observe that $w'' > 0$ by (2.2). Consequently $w > z_0$ on $(\sigma_0, 1)$, where $z_0 = z_0(\sigma) = \alpha(1-\sigma)$.

Now let α be restricted by $0 < \alpha < \beta_0$ and choose β such that $\alpha < \beta < \beta_0$. Let z be the solution of (2.4) which was constructed above, and which of course is defined on the entire interval $[\frac{1}{2}, 1]$. Then $w < z$ and $|w'| < |z'|$ on $(\sigma_0, 1)$ by standard comparison arguments and the fact that Ω is increasing and f is non–decreasing. By the same reasoning w can be continued backward, with $z_0 < w < z$, $|w'| < |z'|$, $w'' \geq 0$, until one reaches $\sigma = 1/2$.

Finally if β_0 is made even smaller if necessary, then $|w'| < |z'| < 1$. This choice of β_0 we now call α_0.

To complete the proof we note some important properties of the solution $w = w_\alpha$.

2. *Uniqueness and continuous dependence on* α. This follows from standard theory of ordinary differential equations applied to the inverse function $\sigma = \sigma(w)$, which satisfies the corresponding equation

$$\Omega'\left(\frac{1}{|\sigma_w|}\right)\sigma_{ww} + \frac{k}{\sigma}\Omega\left(\frac{1}{|\sigma_w|}\right)\sigma_w^3 + \ell f(bw)\sigma_w^3 = 0, \qquad w \geq 0,$$

with $\sigma(0) = 1$, $\sigma_w(0) = -1/\alpha$.

3. *Monotonicity of* $\alpha \mapsto w_\alpha$. As earlier, this is a consequence of standard comparison arguments and the fact that f is non–decreasing. That $w_\alpha(\frac{1}{2})$ has a (finite) limit as $\alpha \to \alpha_0$ follows immediately.

Remark. The condition $|w'| < |z'| < 1$ forces $\alpha_0 < 1$ and $m_0 < 1/2$. In some cases, however, the principal inequality of (B) holds without the restriction $|p| < 1$. In this case, the condition $|Dv| < 1$ is not required in (2.1) and in turn the proof of Proposition 1 can be carried out without the condition $|z'| < 1$. It can then happen that $\beta_0 = \infty$, hence also $\alpha_0 = \infty$ and $m_0 = \infty$. We take up this situation in more detail in Proposition 3 below.

Now define $m(\alpha) = w_\alpha(\frac{1}{2})$, $\alpha \in J = (0, \alpha_0)$. The properties of the function m are given by the following

Proposition 2. *Under the assumptions of Proposition 1, we have*

(i) $m(\alpha) \to 0$ *as* $\alpha \to 0$,
(ii) m *is continuous and strictly increasing on* J,
(iii) $m(\alpha) > \alpha/2$,
(iv) $m_0 = \lim_{\alpha \to \alpha_0} m(\alpha)$.

This follows directly from Proposition 1.

It is convenient in the sequel to have $m_0 = \infty$ in all cases. Indeed, this can always be maintained by replacing $m(\alpha)$ by a new function $\hat{m}(\alpha)$, also defined on J and obeying *(i)–(iv)*, but additionally such that $\hat{m}(\alpha) \geq m(\alpha)$ and $\lim_{\alpha \to \alpha_0} \hat{m}(\alpha) = \infty$. Obviously then, for all $\alpha \in J$ we have

$$\hat{m}(\alpha) \geq w_\alpha(\tfrac{1}{2}), \qquad \alpha \in J. \tag{2.5}$$

In the main proofs of Section 3 it is only Propostion 2 which is needed, together with (2.5). Thus in the sequel we can drop the "hat" from $\hat{m}$ without confusion resulting.

Proposition 3. *Let $\Omega(\infty) = \infty$. Under the assumptions of Proposition* 1*, we have $\alpha_0 = \infty$ if the principal inequality of* (B) *holds for all $p \in \mathbb{R}^2$ and if, moreover, either $f \equiv 0$ for $u \geq 0$ or*

$$\int_1^\infty \frac{ds}{H^{-1}(F(s))} = \infty. \tag{2.6}$$

Proof. First note that $H(\infty) = \infty$ so that the integral in (2.6) is well–defined.

Because of the strengthened form of (B) the condition $|Dv| < 1$ in (2.1) is not needed in the construction of a minorant function v. It is therefore enough to show that problem (2.4) has a solution over the entire interval $[\frac{1}{2}, 1]$ for *all* $\beta > 0$. Suppose not. Then there are $\beta > 0$ and $\bar{\sigma} \in (\frac{1}{2}, 1)$ such that the corresponding local solution z is piecewise C^2 in $(\bar{\sigma}, 1]$ and $z(\sigma) \to \infty$ as $\sigma \to \bar{\sigma}$, see [PS, proof of Lemma 2]. In particular, by [PS, (2.16)] and the fact that $\Omega(\infty) = \infty$, the case $z(\bar{\sigma}) < \infty$, $z'(\bar{\sigma}) = -\infty$ cannot occur at the blow–up point $\bar{\sigma}$.

Also, as in [PS, (2.24)], we have

$$|z'(\sigma)| \leq H^{-1}(\nu F(bz(\sigma))) \qquad \text{for all } \sigma \in (\bar{\sigma}, \rho], \tag{2.7}$$

where $\nu > 1$ and $\rho \in (\bar{\sigma}, 1)$ are appropriate constants depending only on β, b, θ, μ, f and Ω. Integration of (2.7) on $(\bar{\sigma}, \rho]$ gives

$$\int_\gamma^\infty \frac{ds}{H^{-1}(F(s))} \leq \frac{\nu b}{2},$$

where $\gamma = \nu b z(\rho)$. This contradicts (2.6) and completes the proof.

3. Proof of Theorem 1

We begin with two lemmas.

Lemma 5. *Assume Ω satisfies conditions* (Ω1)–(Ω2)*. Let u_1 and u_2 be respective C^1 distribution solutions of*

$$\operatorname{div}\{A(|Du_1|)Du_1)\} - \hat{B}(u_1, Du_1) \leq 0, \qquad u_1 \geq 0, \tag{3.1}$$

and of

$$\operatorname{div}\{A(|Du_2|)Du_2)\} - \hat{B}(u_2, Du_2) \geq 0, \qquad u_2 \geq 0, \tag{3.2}$$

in a bounded domain D, where the function $\hat{B}(u,p)$ is non-decreasing in the variable u and continuously differentiable in the variable p for $p \in \mathbb{R}^2 \backslash \{0\}$. Suppose also that u_1 and u_2 are continuous on $\overline{D}$ and satisfy $|Du_1| + |Du_2| > 0$ there.

Then $u_1 \geq u_2$ on ∂D implies $u_1 \geq u_2$ in D.

This is the weak comparison principle given in [PSZ, Lemma 5] for $D \subset \mathbb{R}^n$ and for $\hat{A}(p) = pA(|p|)$.

Lemma 6. *Assume Ω satisfies conditions* $(\Omega 1)$–$(\Omega 2)$ *and that the function $B(p)$ obeys* (B). *Let u be a solution of* $(1.2)_{\text{left}}$ *in a bounded domain D and let $d > 0$ be such that the set $U_d = \{x \in D \,:\, u(x) > d\}$ is non–empty. Then $\partial K \cap \partial D$ is non–empty for any component K of U_d.*

Proof. Suppose $\partial K \cap \partial D = \emptyset$. Since ∂K and ∂D are compact, then $\text{dist}(\partial K, \partial D) > 0$, and $\overline{K} \subset D$. It then follows from the definition of U_d and K that $u = d$ on ∂K.

We now wish to apply Lemma 5. Here it can be assumed without loss of generality that $D \subset B_R$ for some $R > 0$. Let $\epsilon > 0$ and define

$$u_1 = u_1(y) = d + \int_R^y \Omega^{-1}(\epsilon\, e^{-\mu s})\, ds \quad \text{for } \; y \in [-R,\, R],$$

where μ is the constant in condition (B). One easily checks that

$$\text{div}\{A(|Du_1|)Du_1)\} = \{\Omega(|Du_1|)\}' = -\mu\Omega(|Du_1|) \quad \text{in } D.$$

Let ϵ be taken so small that $|Du_1| < 1$ in D. Then by (B) u_1 obeys (3.1) with $\hat{B}(u,p) = -B(p)$, while also $|Du_1| > 0$ in B_R. Next put $u_2 = u$, so by $(1.2)_{\text{left}}$ the inequality (3.2) holds, again with $\hat{B}(u,p) = -B(p)$. Obviously $u_1 > u_2$ on ∂K. Hence by Lemma 5 with $D = K$ we obtain $u = u_2 \le u_1 = u_1(y)$. Letting $\epsilon \to 0$ then yields $u \le d$ in K, an immediate contradiction.

It almost goes without saying that Lemmas 5 and 6 also hold in arbitrary dimensions $n \ge 2$.

For the reminder of the section we follow [S1], see also [PW, pages 111–117] and [GT, pages 41–44].

Let u be a positive solution of (1.2) in B_R. From Lemma 6, with $D = B_R$, $d = u(0,0)/2$ and K the component of U_d containing $(0,0)$, it follows at once that there is a curve Γ in U_d joining $(0,0)$ and some point $Q \in \partial B_R$. By rotation of coordinates it can be assumed without loss of generality that $Q = (0,\, R)$, see [S1], Figure 1 on page 296] or [PW, Figure 17 on page 112]. As in [S1], see also [PW, pages 112-113], we introduce two ellipses E_1 and E_2, with horizontal major axes and $a = 3R$, $b = R/2$ (so $\theta = a/b = 6$), and with respective centers at $(-5R/2,\, R/2)$ and $(5R/2,\, R/2)$. Let Σ_1 and Σ_2 be corresponding minorant regions, chosen so that $E_1 \cap B_R \subset \Sigma_1$ and $E_2 \cap B_R \subset \Sigma_2$. Clearly $T \equiv \Sigma_1 \cap \Sigma_2 \subset B_R$, and in fact T contains the line segment τ joining the vertices $(-R/2,\, R/2)$ and $(R/2,\, R/2)$ of E_1 and E_2..

We complete the construction by introducing a third ellipse E_3, with vertical major axis and $a = 3R$ and $b = R/2$, and with center at $(0,\, 2R)$. Let Σ_3 be the associated minorant region which is contained in B_R; in particular Σ_3 is bounded above by a proper subsegment $\hat{\tau}$ of τ.

A simple calculation shows that the closed disk $\overline{B_{R/3}}$ is contained in the interior of Σ_3, see [S, Figure 2 on page 297]; in fact the local coordinate σ in Σ_3 satisfies $\sigma \le 0.9542(\approx 2\sqrt{2}/3)$ at all points in $B_{R/3}$. Also, for future use, one finds easily that the local coordinate σ in Σ_1, or in Σ_2, is such that $\sigma \ge (10+\sqrt{3})/12$ on $\hat\tau$.

Let $\hat\alpha$ be defined by $bm(\hat\alpha) = \frac{1}{2}u(0,0)$, and put $\hat w = w_{\hat\alpha}$, the solution of (2.2)–(2.3) with $\alpha = \hat\alpha$. Also let $\sigma_1 : \Sigma_1 \to [\frac{1}{2}, 1]$, $\sigma_2 : \Sigma_2 \to [\frac{1}{2}, 1]$ be, respectively, the local coordinates σ on Σ_1 and Σ_2. Then $v_1 = b\hat w(\sigma_1) : \Sigma_1 \to \mathbb{R}^+$, $v_2 = b\hat w(\sigma_2) : \Sigma_2 \to \mathbb{R}^+$ are minorant functions associated to Σ_1 and Σ_2.

In particular, for points on the straight boundary segment $\xi = a/2$ of Σ_1, it is evident that $v_1 \le b\hat w(\frac{1}{2}) \le bm(\hat\alpha) = \frac{1}{2}u(0,0)$; similarly, on the boundary segment $\xi = a/2$ of Σ_2, we have $v_2 \le \frac{1}{2}u(0,0)$. In turn

$$v_1 \le \tfrac{1}{2}u(0,0) \quad \text{in } \Sigma_1, \qquad v_2 \le \tfrac{1}{2}u(0,0) \quad \text{in } \Sigma_2.$$

Then, following [S1] and using the comparison Lemma 5 with $\hat B(u,p) = B(p) + f(u)$, and with $u_1 = u$ (hence $(1.2)_{\text{right}}$ implies that u_1 satisfies (3.1)) and u_2 equal to either v_1 or v_2 (so by (2.1) u_2 satisfies (3.2)), we get

$$u(x,y) \ge \min\{v_1,\, v_2\} = b\min\{\hat w(\sigma_1), \hat w(\sigma_2)\} \qquad \text{on } \tau. \tag{3.3}$$

Note here that $|Dv_1| > 0$, $|Dv_2| > 0$ so Lemma 5 remains applicable. Moreover, since $\hat w'' \ge 0$ and $\hat w(1) = 0$, $\hat w'(1) = -\hat\alpha$, we find

$$\hat w(\sigma_1),\, \hat w(\sigma_2) \ge \left(1 - \frac{10+\sqrt{3}}{12}\right)\hat\alpha \qquad \text{on } \hat\tau. \tag{3.4}$$

In the same way, let $\hat\gamma$ be such that $m(\hat\gamma) = \frac{1}{12}(2-\sqrt{3})\hat\alpha$, and denote by $\hat\phi$ the solution of (2.2)–(2.3) with $\alpha = \hat\gamma$. Finally, let σ_3 be the local coordinate σ on Σ_3, and consider the minorant function $v_3 = b\hat\phi(\sigma_3) : \Sigma_3 \to \mathbb{R}^+$ associated with Σ_3. Then from (3.3), (3.4), and from Lemma 5 with $u_1 = u$ and $u_2 = v_3$ (and again for $\hat B(u,p) = B(p) + f(u)$), we obtain

$$u(x,y) \ge v_3(x,y) \quad \text{in } \Sigma_3. \tag{3.5}$$

From the main construction above one sees that

$$v_3(x,y) \ge \frac{3-2\sqrt{2}}{3} b\hat\gamma = \frac{3-2\sqrt{2}}{6} R\hat\gamma \quad \text{on } B_{R/3} \tag{3.6}$$

since $b = R/2$. Combining the inequalities (3.5) and (3.6) yields, for $(x,y) \in B_{R/3}$,

$$u(x,y) \ge \frac{3-2\sqrt{2}}{6} Rm^{-1}\left(\frac{2-\sqrt{3}}{12} m^{-1}\left(\frac{u(0,0)}{R}\right)\right) \equiv \Phi_R(u(0,0)),$$

where

$$\Phi_R(t) = \frac{3-2\sqrt{2}}{6} R m^{-1}\left(\frac{2-\sqrt{3}}{12} m^{-1}\left(\frac{t}{R}\right)\right).$$

The existence of the inverse function m^{-1} follows from Proposition 2. By the agreement $m_0 = \infty$ it is evident that m^{-1} maps $[0,\infty)$ strictly monotonically onto $[0,\alpha_0)$. Hence Φ_R is strictly increasing on $[0,\infty)$, and the proof is complete.

Proof of Theorem 2. Define

$$M_R = \begin{cases} \dfrac{3-2\sqrt{2}}{6} R m^{-1}\left(\dfrac{2-\sqrt{3}}{12}\alpha_0\right) & \text{if } \alpha_0 < \infty \\ \infty & \text{if } \alpha_0 = \infty, \end{cases} \tag{3.7}$$

so in particular Φ_R is strictly increasing from $[0,\infty)$ onto $[0, M_R)$, and admits an inverse, say $\Psi_R = \Phi_R^{-1} : [0, M_R) \to [0, t_0)$. The conclusion now follows by exactly the proof of Corollary 1 in [S1].

Theorem 3. *Let the hypotheses of Proposition* 3 *hold. Then* $M_R = \infty$.

Proof. By Proposition 3 we have $\alpha_0 = \infty$, so the conclusion is an immediate consequence of the definition (3.7) of M_R .

Remark. Conditions (1.3) and (2.6) place non-trivial restrictions on the behavior of the function f for u near 0 and u near ∞. For example, if $A(t) = t^{m-2}$ (m–Laplace operator), then $H(t) = mt^m/(m-1)$. Thus a function f with polynomial behavior $f \approx u^p$ near $u = 0$ and $f \approx u^q$ near $u = \infty$ satisfies (1.3), (2.6) only when $p \geq m-1$, $q \leq m-1$.

4. A variational example

Here we consider the Euler equation for the variational problem

$$\delta \int a(x,y,u)\big[G(|Du|) + F(u)\big]dxdy = 0,$$

where the integrands $G(t)$ and $F(u)$ are given at the beginning of the introduction, and $a = a(x,y,u)$ is a positive C^1 function satisfying

$$0 \leq \frac{a_u}{a} \leq \text{Const.}. \tag{4.1}$$

It is checked easily that the Euler equation takes the form

$$\operatorname{div}\{A(|Du|)Du\} = -\frac{Da\cdot Du + a_u|Du|^2}{a}A(|Du|) + f(u) + \frac{a_u}{a}\big[G(|Du|) + F(u)\big]. \tag{4.2}$$

We show this can be put in the form (1.2). First, since we deal always with bounded domains (e.g. B_R), there exists $\hat{\mu} > 0$ such that

$$0 \le \frac{a_u}{a} \le \hat{\mu}, \qquad \frac{|Da|}{a} \le \hat{\mu},$$

while moreover, since f and Ω are non-decreasing,

$$F(u) \le uf(u), \qquad G(t) \le t\Omega(t).$$

Hence, noting particularly that $a_u \ge 0$, we see that, as required,

$$-B(|Du|) \le \operatorname{div}\{A(|Du|)Du\} \le \hat{\mu}\Omega(|Du|) + \hat{f}(u), \tag{4.3}$$

where

$$B(p) = \hat{\mu}(1 + |p|)\Omega(|p|), \qquad \hat{f}(u) = (1 + \hat{\mu}\, u)f(u).$$

Here $B(p) \le 2\hat{\mu}\Omega(|p|)$ when $|p| < 1$, so (B) holds with $\mu = 2\hat{\mu}$.

Since $\hat{f}$ obviously satisfies (F) whenever f does, and since equally clearly

$$\int_0^1 \frac{ds}{H^{-1}(\hat{F}(s))} = \infty$$

if (1.3) holds, we have the following result.

Theorem 4. *Let the function* Ω *satisfy* (Ω1)–(Ω4), *and assume* f *obeys* (F). *Suppose also either* $f \equiv 0$ *on* $[0, \delta]$, $\delta > 0$, *or* $f > 0$ *on* $\mathbb{R}^+$ *and*

$$\int_0^1 \frac{ds}{H^{-1}(F(s))} = \infty. \tag{1.3}$$

Then for every $R > 0$ *there is a strictly increasing continuous function* $\Phi_R : [0, \infty) \to [0, \infty)$, *with* $\Phi_R(0) = 0$, *such that any* C^1 *distribution solution* u *of the Euler equation* (4.2) *in the disk* B_R *satisfies the Harnack condition*

$$u(x, y) \ge \Phi_R(u(0, 0)) \qquad \text{for all } (x, y) \in B_{R/3}.$$

It is obvious that Theorem 2 also holds for equation (4.2).

In conclusion, it is not hard to see that in order to have $M_R = \infty$ one needs the condition (4.1) to hold in the stronger form

$$(1 + u)\frac{a_u}{a} \le \text{Const.}$$

(so that one can take $\hat{f}(u) = \text{Const.}\, f(u)$), and also that either $f \equiv 0$ or (2.6) is satisfied. [Here one uses the explicit form of the right hand side of (4.3), from which it follows that the restriction $|Dv| < 1$ is *not* required for the construction of the minorant function v; see the Remark before Proposition 2.]

Acknowledgement. This work is partially supported by CNR, Progetto Strategico *Modelli e Metodi per la Matematica e l'Ingegneria*, by the Italian *Ministero della Università e della Ricerca Scientifica e Tecnologica* under the auspices of *Gruppo Nazionale di Analisi Funzionale e sue Applicazioni* of the CNR and by the Project *Metodi Variazionali ed Equazioni Differenziali Non Lineari.*

References

[BFP] Benci V., D. Fortunato and L. Pisani, *Solitons like solutions of a Lorentz invariant equation in dimension* 3, Rev. Math. Phys. **10** (1998), 315–344.

[D] Diaz J.I., *Nonlinear partial differential equations and free boundaries*, Research Notes in Math. **106**, 1985.

[GT] Gilbarg, D. and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition Springer-Verlag, 1983.

[G] Gregori, G., *Generalized solutions for a class of non–uniformly elliptic equations in divergence form*, Comm. Part. Diff. Eqs. **22** (1997), 581–617.

[K] Keller J., *On solutions of* $\Delta u = f(u)$, Comm. Pure Applied Math. **10** (1957), 503–510.

[L] Lieberman, G.M., *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Diff. Eqs. **16** (1991), 311–361.

[O] Osserman R., *On the inequality* $\Delta u \geq f(u)$, Pac. J. Math. **7** (1957), 1641–1647.

[PS] Pucci P. and J. Serrin, *A note on the strong maximum principle for elliptic differential inequalities*, J. Math. Pures Appl. **79** (2000), 57–71.

[PSZ] Pucci P., J. Serrin and H. Zou, *A strong maximum principle and a compact support principle for singular elliptic inequalities*, J. Math. Pures Appl. **78** (1999), 769–789.

[PW] Protter, M.H. and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice–Hall, 1967.

[S1] Serrin, J., *On the Harnack inequality for linear elliptic equations*, J. d'Analyse Math. **4** (1954–56), 292–308.

[S2] Serrin, J., *Local behavior of solutions of quasi–linear equations*, Acta Mathematica **111** (1964), 247–302.

[T] Trudinger, N.S., *Harnack inequalities for nonuniformly elliptic divergence structure equations*, Inventiones Math. **64** (1981), 517–531.

[V] Vazquez J.–L., *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.