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ASYMPTOTIC ESTIMATES FOR A NONSTANDARD SECOND ORDER DIFFERENTIAL EQUATION

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§1. Introduction.

We are concerned with the asymptotic behavior at infinity of solutions of the linear differential equation

$$y'' + 2b\, t^{\alpha-1} y' + c\, t^{2\beta-2} y = 0, \tag{1.1}$$

where α, $\beta \in I\!R$ and b, c are non–zero real constants. Such problems arise for example in the study of radial solutions of the linear elliptic equation $\Delta u + c|x|^{2\beta-2}u = 0$ in n dimensions, when $u(r) = u(|x|)$ is a solution of

$$u''(r) + \frac{n-1}{r} u'(r) + c\, r^{2\beta-2} u(r) = 0, \qquad r > 0.$$

Another case of importance is that of a damped harmonic oscillator, when $\beta = c = 1$ and $b > 0$. More generally we may think of (1.1) as a linear differential equation with a highly irregular singular point at ∞.

In an earlier paper [2] we showed that the rather unexpected results indicated in [4] for the special case $\beta = 1$ have the following more general manifestation: *the* (α, β) *plane* *is divided into sectors by the set of rays from the origin* $(0,0)$ *given by*

$$(1.2)\qquad \begin{cases} \beta<0, & \alpha=0 \\ \beta=s_N\alpha, & \alpha>0, \quad N=0,1,\dots, \quad s_N=1-1/2(N+1), \\ \beta=\alpha, & \alpha>0 \\ \beta=t_N\alpha, & \alpha>0, \quad N=0,1,\dots, \quad t_N=1+1/(2N+1)=1/s_N, \\ \beta>0, & \alpha=0 \\ \beta=0, & \alpha<0 \end{cases}$$

which represent lines of change in the asymptotic representation of y *and* y'*.*

The concept of a change in asymptotic behavior of soultions as (α,β) crosses one of the rays (1.2) is sufficiently delicate that we indicate here precisely the meaning which is intended. This is somewhat easier to state for the non–oscillatory case, so we consider this first. Here, corresponding to any open sector S in $I\!R^2$ bounded by a pair of consecutive rays (1.2), there is a positive real analytic function $\phi_S(t)=\phi_S(t;\alpha,\beta)$, $(\alpha,\beta)\in S$, $t>0$, having the property that

$$(1.3)\qquad \frac{y(t)}{\phi_S(t)} \to \text{ finite limit} = \ell \qquad \text{as } t\to\infty$$

for any (non–oscillatory) solution y of (1.1) for which $(\alpha,\beta)\in S$. Explicit expressions for the asymptotic functions ϕ_S are given in [2].

Turning to the oscillatory case, we note to begin with that this can occur only when $c>0$, and only for (α,β) in the parameter region defined by

$$\beta \geq \max\{0,\alpha\},$$

see [2]. Then, corresponding to any sector S in this region, there are *two* real analytic functions

$$(1.4)\qquad \phi_S(t)=\phi_S(t;\alpha,\beta), \quad \psi_S(t)=\psi_S(t;\alpha,\beta), \quad (\alpha,\beta)\in S, \quad t>0,$$

such that

$$(1.5)\qquad \frac{y(t)}{\phi_S(t)} = [A\cos(\psi_S(t)+\theta)]\cdot[1+o(1)]$$

for any real solution y of (1.1) for which $(\alpha,\beta)\in S$, where A, θ are appropriate real constants depending on the solution.

The function ϕ_S and ψ_S depend continuously on $(\alpha, \beta) \in S$, and also on b, c for b, $c \neq 0$. Moreover, they have *continuous extensions* in the variables α, β to the *closure* of S in $\mathbb{R}^2 \backslash \{(0,0\}$. The extended functions obey the following *transition formulas* on rays R of the form (1.2), separating a pair of adjacent sectors S and S':

Non–oscillatory case. For $(\alpha, \beta) \in R$

$$\phi_{S'}(t; \alpha, \beta) = t^\lambda \phi_S(t; \alpha, \beta), \quad t > 0, \tag{1.6}$$

where λ is a non–zero constant, depending only on b, c and the ray R.

Oscillatory case. For $(\alpha, \beta) \in R$

$$\phi_{S'}(t; \alpha, \beta) = \phi_S(t; \alpha, \beta), \quad \psi_{S'}(t; \alpha, \beta) = \psi_S(t; \alpha, \beta) + \mu \log t, \quad t > 0, \tag{1.7}$$

where μ is a non–zero real constant, again depending only on b, c and the ray R.

The formulas (1.6), (1.7) graphically illustrate the jump discontinuity in asymptotic behavior which occurs as one crosses a ray (1.2). Note that in the oscillatory case the *amplitude* function ϕ *does not change* across rays, there being only a *phase shift* in the amount $\mu \log t$.

In this paper we shall refine the above results to obtain the asymptotic behavior as $t \to \infty$ for the first and second derivatives of solutions of (1.1). In particular we show in the non–oscillatory case that

$$\frac{y'(t)}{\phi'(t)} \to \ell \quad \text{as } t \to \infty, \tag{1.8}$$

with the exception of parameter values (α, β) on the following rays

$$\alpha = \beta > 0; \quad \alpha = 0, \ \beta < 0; \quad \beta = 0, \ \alpha < 0.$$

In (1.8) the function ϕ is given by

$$\phi(t) = \begin{cases} \phi_S(t; \alpha, \beta) & \text{if } (\alpha, \beta) \in S, \\ t^\lambda \phi_S(t; \alpha, \beta) & \text{if } (\alpha, \beta) \in R, \end{cases}$$

where in the second case S is one of the two sectors adjacent to R.

On the other hand, in the *oscillatory case* we show that

$$\frac{[t^{1-\beta} y'(t)]^2 + c\, y^2(t)}{\phi^2(t)} \to c\, A^2 \quad \text{as } t \to \infty,$$

again with the exception of the rays

$$\alpha = \beta > 0; \quad \alpha = 0,\ \beta < 0; \quad \beta = 0,\ \alpha < 0.$$

The behavior of the second derivatives is treated in Section 6.

For the non–oscillatory case our proofs are based on the Sturmian form of (1.1), namely

$$(p\,y')' + c\,p\,t^{2\beta-2}\,y = 0, \tag{1.9}$$

where

$$p = p(t) = \begin{cases} \exp\left(\dfrac{2b}{\alpha}\,t^{\alpha}\right), & \alpha \neq 0 \\ \\ t^{b}, & \alpha = 0. \end{cases} \tag{1.10}$$

For the oscillatory case we work directly with the fundamental matrix $Z(t)$ related to equation (1.1).

The origin $\alpha = 0$, $\beta = 0$, corresponds to the Euler equation $t^2y'' + 2b\,t\,y' + c\,y = 0$, whose treatment is standard and which is best set aside from the main considerations of the paper. *Thus in what follows we assume always that* $(\alpha, \beta) \neq (0, 0)$.

§2. Asymptotic behavior when $\alpha > \beta$ and $\alpha > 0$.

This case is best divided into the two subcases $\alpha < 2\beta$ and $\alpha > 2\beta$. The treatment of asymptotic behavior on the rays (1.2) themselves is slightly different than for the open sectors between the rays. For simplicity in this and the next two sections, we shall therefore only treat values (α, β) in the interior of the sectors. The discussion when (α, β) belongs to a ray will be given later at the end of Section 5.

2.1 The case $\alpha < 2\beta$ and $b > 0$. Here by (3.8) of [2], we have

$$\phi(t) = \phi_N(t) = \exp\left(\sum_{1}^{N} \lambda_k \int_1^t s^{\alpha-1+2k(\beta-\alpha)}ds\right) = \exp\Sigma(t), \qquad t > 0, \tag{2.1}$$

where all the coefficients λ_k are real and non–zero. It follows from (1.9) that

$$\frac{y'(t)}{\phi'(t)} = -\frac{c}{p(t)\phi'(t)}\int^t p(s)s^{2\beta-2}y(s)ds. \tag{2.2}$$

By (2.1) we get

$$\Sigma(t) = \frac{\lambda_1}{2\beta - \alpha}\, t^{2\beta-\alpha}\,[1+o(1)],$$

$$\frac{\phi'(t)}{\phi(t)} = \Sigma'(t) = \lambda_1 t^{2\beta-\alpha-1}\,[1+o(1)], \tag{2.3}$$

and so from (1.8)

$$p(t)\phi'(t) \;\to\; \infty \qquad \text{as } t \to \infty. \tag{2.4}$$

We now apply L'Hospital's rule to the right hand side of (2.2) in order to determine its limit as $t \to \infty$. Suppose first that in (1.3)

$$\frac{y(t)}{\phi(t)} \;\to\; \ell \neq 0 \qquad \text{as } t \to \infty. \tag{2.5}$$

Then the integral in (2.2) is divergent to $\pm\infty$ as $t \to \infty$, since $\alpha < 2\beta$. L'Hospital's rule therefore gives

$$\lim_{t\to\infty} \frac{y'(t)}{\phi'(t)} = -\lim_{t\to\infty} \frac{c\, p(t)\, t^{2\beta-2}\, y(t)}{p(t)\phi''(t) + p'(t)\phi'(t)}, \tag{2.6}$$

provided the right hand limit exists.

To investigate this, we observe first that $p' = 2b\, t^{\alpha-1}\, p$. It is therefore enough to consider the behavior as $t \to \infty$ of the function

$$I = \frac{\phi'' + 2b\, t^{\alpha-1}\phi'}{t^{2\beta-2} y} = \left(\frac{\phi''}{\phi} + 2b\, t^{\alpha-1}\frac{\phi'}{\phi}\right)\frac{1}{t^{2\beta-2}} \cdot \frac{\phi}{y}. \tag{2.7}$$

From (2.3) we have

$$\frac{\phi''(t)}{\phi(t)} = \Sigma''(t) + (\Sigma'(t))^2 = \lambda_1^2\, t^{2(2\beta-\alpha-1)}\,[1+o(1)], \tag{2.8}$$

since $\alpha < 2\beta$. Thus, using (2.3),

$$\frac{\phi''}{\phi} + 2b\, t^{\alpha-1}\,\frac{\phi'}{\phi} = 2b\, \lambda_1\, t^{2\beta-2}\,[1+o(1)]$$

since $\alpha > \beta$. Consequently

$$\lim_{t\to\infty} I(t) = \frac{2b\lambda_1}{\ell}$$

by (2.5).

Hence the limit on the right hand side of (2.6) exists, and so in turn

$$\frac{y'(t)}{\phi'(t)} \to \text{finite limit;}$$

by (2.5) and by L'Hospital's rule this limit must in fact be ℓ, as required.

Next suppose that $\ell = 0$ in (1.3). Fix $\varepsilon > 0$. Then $|y(t)/\phi(t)| < \varepsilon$ for all t sufficiently large, say $t \geq T$. Hence from (2.2) and (2.4)

$$\begin{aligned}\left|\frac{y'(t)}{\phi'(t)}\right| &\leq \frac{|c|}{p|\phi'|}\left|\left(\int^T + \int_T^t\right) p(s)s^{2\beta-2}y(s)ds\right| \\ &\leq \varepsilon + \varepsilon\,\frac{|c|}{p(t)|\phi'(t)|}\int_T^t p(s)s^{2\beta-2}\phi(s)ds\end{aligned}$$

for suitably large t. The use of L'Hospital's rule as in the previous calculation then gives

$$\limsup_{t\to\infty}\left|\frac{y'(t)}{\phi'(t)}\right| \leq \varepsilon + \varepsilon\,\frac{|c|}{2b|\lambda_1|}.$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$\lim_{t\to\infty}\frac{y'(t)}{\phi'(t)} = 0,$$

as required.

2.2 The case $\alpha < 2\beta$ and $b < 0$. Here, instead of (2.1), we have by (3.9) of [2],

$$\phi(t) = \phi_N(t) = t^{1-\alpha}\exp\left(-\frac{2b}{\alpha}\left[t^\alpha - 1\right] + \Sigma(t)\right),$$

where the function $\Sigma(t)$ was defined in (2.1). Corresponding to (2.3) we have, as $t \to \infty$,

$$\frac{\phi'(t)}{\phi(t)} = (1-\alpha)\,t^{-1} - 2b\,t^{\alpha-1} + \lambda_1\,t^{2\beta-\alpha-1}\left[1 + o(1)\right], \tag{2.9}$$

and then, after a short calculation,

$$\frac{\phi''(t)}{\phi(t)} = \left[\frac{\phi'(t)}{\phi(t)}\right]' + \left[\frac{\phi'(t)}{\phi(t)}\right]^2 = 4b^2t^{2\alpha-2} - 4b\lambda_1 t^{2\beta-2}\,[1+o(1)], \tag{2.10}$$

since $\beta < \alpha < 2\beta$.

In order to apply L'Hospital's rule, we investigate the term $p\,\phi'$ in (2.2). A simple calculation gives

$$p(t)\phi'(t) = 2|b|\exp\left(\frac{\lambda_1}{2\beta-\alpha}\,t^{2\beta-\alpha}\,[1+o(1)]\right) \longrightarrow \begin{cases} 0 & \text{if } \lambda_1 < 0 \\ \infty & \text{if } \lambda_1 > 0 \end{cases} \tag{2.11}$$

as $t \to \infty$.

We first consider the case when $\lambda_1 > 0$. Let (2.5) hold, that is $\ell \neq 0$ in (1.3). Then L'Hospital's rule applies to the right hand side of (2.2), since the integral also diverges. As before, it is clear that we must investigate the function I given in (2.7). By (2.9) and (2.10)

$$\frac{\phi''}{\phi} + 2b\,t^{\alpha-1}\,\frac{\phi'}{\phi} = -2b\lambda_1\,t^{2\beta-2}\,[1+o(1)],$$

since $\alpha < 2\beta$. Consequently

$$\lim_{t\to\infty} I(t) = -\frac{2b\lambda_1}{\ell}.$$

The limit on the right hand side of (2.6) exists, and so

$$\lim_{t\to\infty}\frac{y'(t)}{\phi'(t)} = \ell.$$

The case $\ell = 0$ in (1.3) is treated as in subsection *2.1*.

We now consider the case when $\lambda_1 < 0$. Here the integral on the right hand side of (2.2) converges and so (2.2) can be written in the form

$$\frac{y'(t)}{\phi'(t)} = \frac{c}{p(t)\phi'(t)}\left(\int_t^\infty p(s)s^{2\beta-2}y(s)ds + d\right), \tag{2.12}$$

where d is a constant.

If $d \neq 0$ then y'/ϕ' tends to $\pm\infty$ as $t \to \infty$. In turn, since $\phi(t) \to \infty$, we obtain by integration

$$\lim_{t\to\infty}\frac{y(t)}{\phi(t)} = \infty,$$

which is a contraction. Hence $d = 0$ in (2.12).

We can now apply L'Hospital's rule directly to (2.12), and the remaining discussion is unchanged from the previous case $\lambda_1 > 0$.

It remains to treat

2.3. The case $\alpha > 2\beta$. Suppose first that $b > 0$. Then by (3.8) of [2], with $N = 0$,

$$\phi = \phi_0(t) = 1,$$

where we recall the agreement that $\sum_1^0 = 0$, see footnote 1 in [2]. Here the ratio y/ϕ tends to a finite limit as $t \to \infty$ and so (1.3) takes the simple form

$$y(t) \to \ell \qquad \text{as } t \to \infty.$$

Actually in the present case it is convenient to choose a slightly different function ϕ, namely

$$\phi(t) = 1 - \frac{c}{2b(2\beta - \alpha)} t^{2\beta-\alpha}.$$

Of course, even with this choice we still have

$$\frac{y(t)}{\phi(t)} \to \ell \qquad \text{as } t \to \infty.$$

Now by (1.9)

$$y'(t) = -\frac{c}{p(t)} \int^t p(s) s^{2\beta-2} y(s) ds, \tag{2.13}$$

where $p(t) \to \infty$ as $t \to \infty$ since $b > 0$. If $\ell \neq 0$ then L'Hospital's rule gives

$$\frac{y'(t)}{\phi'(t)} \to \ell \qquad \text{as } t \to \infty. \tag{2.14}$$

When $\ell = 0$ the argument at the end of case *2.1* shows that (2.14) continues to hold, with $\ell = 0$.

It remains to consider $b < 0$. Here by (3.9) of [2]

$$\phi(t) = \phi_0(t) = t^{1-\alpha} \exp\left(\frac{2|b|}{\alpha} [t^\alpha - 1]\right),$$

and an easy calculation yields $p(t)\phi'(t) \to 2|b|\exp(-2|b|/\alpha) > 0$ as $t \to \infty$. Moreover the integral in (2.2) converges because of (1.3), (1.10) and the fact that $\alpha > 2\beta$. Thus finally

$$\frac{y'(t)}{\phi'(t)} \to \text{limit} = \ell \qquad \text{as } t \to \infty.$$

§3. Asymptotic behavior when $\beta > \alpha > 0$.

This case is best divided into the two subcases $c < 0$ and $c > 0$.

3.1. The non–oscillatory case $c < 0$. By (3.13) in [2] we have

$$\phi(t) = \hat{\phi}_N(t) = t^{(1-\beta)/2} \exp\left(-\frac{b}{\alpha}\,[t^\alpha - 1] + \psi(t)\right), \tag{3.1}$$

where

$$\psi(t) = \psi_N(t) = \sqrt{|c|}\,\sum_0^N \mu_k \int_1^t s^{\beta-1+2k(\alpha-\beta)}ds, \qquad \mu_0 = 1.$$

First one checks, for the formula (2.2), that $p\,\phi'$ tends to ∞ as $t \to \infty$ and that, when $\ell \neq 0$ in (1.3), the integral diverges as $t \to \infty$.

Consequently L'Hospital's rule applies. Proceeding as in subsection *2.1*, we find after a straightforward calculation that the function I in (2.7) satisfies

$$\lim_{t\to\infty} I(t) = \frac{|c|}{\ell}, \qquad \ell \neq 0,$$

since $\beta > \alpha > 0$. Hence from (2.2)

$$\lim_{t\to\infty} \frac{y'(t)}{\phi'(t)} = \ell.$$

The case $\ell = 0$ can be treated in the usual way.

3.2. The oscillatory case $c > 0$. The previous calculations were based on the formula (2.2). It is more convenient in the oscillatory case to work directly with the fundamental matrix $Z(t)$ given by (2.9) in [2]. The function $\lambda(t)$ here has the form, as $t \to \infty$,

$$\lambda(t) = i\sqrt{c}\,M(t) + O(t^{-1-\beta+\alpha}),$$

where $M(t) = M_N(t) = \sum_0^N \mu_k\, t^{\beta-1+2k(\alpha-\beta)} = t^{\beta-1}[1 + o(1)]$ is given in (3.10) of [2].

Now for appropriate constants A and θ we have, as in (1.5),

$$\frac{y(t)}{\phi(t)} = A\cos(\psi(t) + \theta)\cdot[1 + o(1)] \qquad \text{as } t \to \infty, \tag{3.2}$$

where, by (3.14) of [2],

$$\phi(t) = t^{(1-\beta)/2} \exp\left(-\frac{b}{\alpha}\,[t^\alpha - 1]\right).$$

In fact (3.2) is just the linear combination, with coefficients $\frac{1}{2}Ae^{i\theta}$ and $\frac{1}{2}Ae^{-i\theta}$, of the two elements on the first row of the fundamental matrix $Z(t)$. Applying the same operation to the two elements in the second row of $Z(t)$ yields, as $t \to \infty$,

$$\begin{aligned}\frac{y'(t)}{\phi(t)} \quad &= \quad -b\,t^{\alpha-1}A\cos(\psi(t)+\theta)\cdot[1+o(1)] \\ &\qquad +\sqrt{c}\,t^{\beta-1}A\sin(\psi(t)+\theta)\cdot[1+o(1)]+O(t^{-1-\beta+\alpha}).\end{aligned}$$

This formula can be expressed more usefully in the form

$$\frac{[t^{1-\beta}y'(t)]^2+c\,y^2(t)}{\phi^2(t)} = c\,A^2[1+o(1)].$$

It should be observed that the oscillatory case does not have asymptotic formulas for y' corresponding to those of the non–oscillatory case.

§4. Asymptotic behavior when $\alpha < 0$.

4.1 The case $\beta > 0$ and $c < 0$. Here by (3.18) of [2] we have

$$\phi(t) = t^{(1-\beta)/2}\exp\left(\frac{\sqrt{|c|}}{\beta}\,[t^{\beta}-1]\right). \tag{4.1}$$

This case is completely analogous to subsection *3.1*, and we get

$$\frac{y'(t)}{\phi'(t)} \to \ell \qquad \text{as } t \to \infty.$$

4.2. The oscillatory case $\beta > 0$ and $c > 0$. As before it is more convenient in the oscillatory case to work directly with the fundamental matrix $Z(t)$. Here the function $\lambda(t)$ is given by

$$\lambda(t) = i\sqrt{c}\,t^{\beta-1}+O(t^{-1-\beta+\alpha}) \qquad \text{as } t \to \infty.$$

Hence this case is exactly the same as subsection *3.2*, except that because $\alpha < 0$ the function ϕ now has the form

$$\phi(t) = t^{(1-\beta)/2}.$$

4.3. The case $\beta < 0$. From (4.2) of [2] one has immediately

$$\frac{y(t)}{t} \to \ell \quad \text{and} \quad y'(t) \to \ell \quad \text{as } t \to \infty.$$

§5. Behavior on the rays (1.2).

As shown in [2], in the non–oscillatory case the asymptotic behavior of a solution y of (1.1) for (α, β) belonging to a ray R of type (1.2) is governed by the function

$$\phi_R(t) = t^\lambda \phi(t), \tag{5.1}$$

where ϕ is one of the two functions corresponding to the adjacent sectors S and S' whose common boundary is R. Moreover λ is a non–zero constant depending only on b, c and R.

Note: Formula (5.1) *applies on all rays* (1.2) *with the exception of the three special ones*

$$\beta = \alpha,\ \alpha > 0; \qquad \alpha = 0,\ \beta < 0; \qquad \beta = 0,\ \alpha < 0;$$

the following considerations consequently do not apply in these cases.

For the functions ϕ_R in (5.1) the calculations of the previous sections carry over almost unchanged. We indicate the details only when the ray R lies in the sector $\beta < \alpha \leq 2\beta$.

First for $\beta < \alpha < 2\beta$ and $b > 0$, the formula (2.3) becomes

$$\frac{\phi_R'}{\phi_R} = \frac{\lambda}{t} + \frac{\phi'}{\phi} = \frac{\lambda}{t} + \Sigma' = \lambda_1\, t^{2\beta-\alpha-1}\,[1 + o(1)], \tag{5.2}$$

which is exactly as before, while (2.8) takes the form

$$\frac{\phi_R''}{\phi_R} = -\frac{\lambda}{t^2} + \Sigma'' + \left(\frac{\lambda}{t} + \Sigma'\right)^2 = \lambda_1^2\, t^{2(2\beta-\alpha-1)}\,[1 + o(1)], \tag{5.3}$$

again exactly as before. The rest of the argument is now unchanged, yielding

$$\lim_{t\to\infty} \frac{y'(t)}{\phi_R'(t)} = \lim_{t\to\infty} \frac{y(t)}{\phi_R(t)} = \ell.$$

The case $b < 0$ can be treated as above.

For the ray $\alpha = 2\beta$ and $b > 0$, we use (3.4) of [2] with $N = 0$ and the case of equality. Consequently by (2.9) of [2] we obtain

$$\phi_R(t) = t^{\lambda_1}, \qquad \lambda_1 = -c/2b.$$

We claim that

$$\lim_{t\to\infty} \frac{y'(t)}{\phi_R'(t)} = \lim_{t\to\infty} \frac{y'(t)}{\lambda_1 t^{\lambda_1 - 1}} = \ell.$$

Indeed by (1.9)

$$\frac{y'(t)}{t^{\lambda_1-1}} = -\frac{c}{t^{\lambda_1-1}p(t)}\int^t p(s)s^{\alpha-2}y(s)ds,$$

where $t^{\lambda_1-1}p(t) \to \infty$ as $t \to \infty$ since $b > 0$. When $\ell \neq 0$ the integral diverges as $t \to \infty$ and L'Hospital's rule gives

$$\frac{y'(t)}{t^{\lambda_1-1}} \to -\frac{c}{2b}\ell,$$

as claimed. The case $b < 0$ is treated exactly as in subsection *2.3*.

§6. Second derivative behavior.

We determine here the asymptotic behavior of y'' for all non–oscillatory cases in the parameter range $2\beta > \max\{0, \alpha\}$, $\alpha \neq \beta$, and in fact show in these cases that

$$\frac{y''(t)}{\phi''(t)} \to \ell \qquad \text{as } t \to \infty. \tag{6.1}$$

For this purpose we write, using the equation (1.1),

$$\begin{aligned} \frac{y''}{\phi''} &= \frac{\phi}{\phi''}\left\{-2b\,t^{\alpha-1}\frac{\phi'}{\phi}\cdot\frac{y'}{\phi'} - c\,t^{2\beta-2}\,\frac{y}{\phi}\right\} \\ &= -2b\,\ell\,t^{\alpha-1}\frac{\phi}{\phi''}\left\{\frac{\phi'}{\phi} + \frac{c}{2b}\,t^{2\beta-\alpha-1}\right\}[1+o(1)]. \end{aligned} \tag{6.2}$$

When $\beta < \alpha < 2\beta$ and $b > 0$, we get from (2.1)

$$\frac{\phi'}{\phi} = \lambda_1 t^{2\beta-\alpha-1} + \lambda_2\,t^{4\beta-3\alpha-1}[1+o(1)],$$

while from (2.8)

$$\frac{\phi''}{\phi} = \lambda_1^2\,t^{4\beta-2\alpha-2}\,[1+o(1)].$$

Recalling that $\lambda_1 = -c/2b$, and inserting the above estimates into (6.2), we have

$$\frac{y''}{\phi''} \to -2b\,\frac{\lambda_2}{\lambda_1^2}\,\ell = \ell \qquad \text{as } t \to \infty.$$

When $\beta < \alpha < 2\beta$ and $b < 0$, we get from (2.9) and (2.10)

$$\frac{\phi'}{\phi} = -2b\, t^{\alpha-1}\, [1 + o(1)],$$

$$\frac{\phi''}{\phi} = 4b^2 t^{2\alpha-2}\, [1 + o(1)],$$

and the required conclusion follows at once using (6.2).

Finally, when $\beta > \max\{0,\, \alpha\}$ and $c < 0$, we get from (3.1) and (4.1)

$$\frac{\phi'}{\phi} = \sqrt{|c|}\, t^{\beta-1}\, [1 + o(1)],$$

$$\frac{\phi''}{\phi} = \left(\frac{\phi'}{\phi}\right)' + \left(\frac{\phi'}{\phi}\right)^2 = |c|\, t^{2\beta-2}\, [1 + o(1)],$$

and (6.1) follows again.

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