Asymptotic Stability for Non–autonomous Damped Wave Systems

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Abstract

We study the asymptotic stability, as time tends to infinity, of solutions of dissipative wave systems with time dependent nonlinear damping and subject to the action of strongly nonlinear potential energies. In the final part of the paper we also consider the case of polyharmonic wave equations and higher order dissipation terms.

1. Introduction

The problem of asymptotic stability for second order ordinary differential equations is well–known in the literature, even in the non–autonomous case. Recently, several authors have extended part of this work to abstract evolutionary equations, as well as to the case of damped wave equations [6,8,9,11]. Here we present an approach which systematically treats this class of problems. We shall concentrate our attention particularly on nonlinear wave systems with non–autonomous and unbounded dissipation.

Consider specifically the nonlinear damped wave system with Dirichlet data

$$\begin{cases} u_{tt} - \Delta u + Q(t,x,u,u_t) + f(x,u) = 0, & (t,x) \in J \times \Omega, \\ u(t,x) = 0 \qquad (t,x) \in J \times \partial\Omega, \end{cases} \tag{1.1}$$

where $J = [0,\infty)$ and Ω is a bounded open subset of $\mathbb{R}^n$. The values of u are taken in $\mathbb{R}^N$, $N \geq 1$, and

$$Q \in C(J \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N), \qquad f \in C(\Omega \times \mathbb{R}^N \to \mathbb{R}^N).$$

The function Q represents a *nonlinear damping*, so that

$$\big(Q(t,x,u,v),v\big) \geq 0 \quad \text{for all arguments } t,x,u,v, \tag{1.2}$$

where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^N$; moreover f is a *restoring force* derivable from a potential F, that is,

$$(f(x,u),u) \geq 0 \tag{1.3}$$

and

$$f(x,u) = \frac{\partial F}{\partial u}(x,u), \tag{1.4}$$

where $F \in C^1(\Omega \times \mathbb{R}^N \to \mathbb{R}_0^+)$, $F(x,0)=0$.

Problem (1.1) is well–known in a variety of models in mathematical physics, e.g., elastic vibrations in a dissipative media, the telegraphic equation, the damped Klein–Gordon equation. These are all in the scalar case $N=1$, where, it should be noted, the existence of a potential F is redundant, since one can simply integrate the force $f(x,u)$ with respect to u. A canonical example of the type of functions f and Q which we contemplate here is given by the system

$$u_{tt} - \Delta u + A(t)|u_t|^{q-2}u_t + V(x)|u|^{p-2}u = 0, \tag{1.5}$$

where $A \in C(J \to \mathbb{R}_0^+)$, $V \in C(\overline{\Omega} \to \mathbb{R}_0^+)$, the exponents p, q satisfy

$$1 < p < \infty, \qquad 1 < q \leq \max\{p,\ 2n/(n-2)\},$$

($1 < q < \infty$ if $n = 1, 2$) and $F(x,u) = V(x)|u|^p/p$. Even for systems such as this, our results improve on earlier work (e.g., in [11], where the scalar case of (1.5) is treated, it was required that $2 \leq p$, $q \leq 2(n-1)/(n-2)$, see Section 5).

In the context of problem (1.1) the question of asymptotic stability is best considered by means of the natural energy associated with the solutions of the system, namely,

$$Eu(t) = \tfrac{1}{2}\int_\Omega \{|u_t(t,x)|^2 + |Du(t,x)|^2\}dx + \int_\Omega F(x,u(t,x))dx.$$

In particular, the rest field $u(t,x) \equiv 0$ will be called *asymptotically stable* in the mean, or simply *asymptotically stable*, if and only if

$$\lim_{t\to\infty} Eu(t) = 0 \qquad \text{for all solutions } u = u(t,x) \text{ of (1.1).}$$

In this formulation we have tacitly assumed that the solutions in question are classical, but for an adequate and useful theory one must actually consider solutions in a wider class of functions. Indeed, one of the goals of the paper is to formulate a rational definition of solution for (1.1) which is independent of detailed properties of the functions f and Q, and at the same time provides a useful framework for the study of asymptotic stability. This we do in Section 2.

Sections 3 and 4 are devoted to our two main asymptotic stability results, Theorems 3.1 and 4.1. Both of these are based on the a priori existence of a suitable auxiliary function $k = k(t)$. Section 5 presents specific applications in which the function k is chosen in explicit form. Further applications to more general problems than (1.1) are given in Sections 6 and 7. The system discussed in Section 7 also involves higher order dissipation terms of the type introduced by [1, 2, 4, 5, 17].

The primary consideration of the paper is the asymptotic stability of solutions of (1.1). Accordingly, we shall not be concerned with the problem of existence of solutions. This problem has a large literature, see for example [6, 7, 10] and the references cited therein, and has also been treated recently by Zhu [18] under hypotheses closely related to, but somewhat stronger than, (H1)–(H3) below. The reader is referred to this literature for further details.

The linear case

An important special case of (1.5) occurs when $p = q = 2$ and $A(t) = a(t)t^\alpha$, $\alpha \in \mathbb{R}$, so that (1.5) takes the form

$$u_{tt} - \Delta u + a(t)\, t^\alpha u_t + V(x)u = 0, \tag{1.6}$$

(if $\alpha < 0$ we restrict the interval J to the set $t \geq 1$). For simplicity we also suppose $N = 1$, and assume that

$$1/C \leq a(t) \leq C \qquad \text{for } t \in J,$$

where C is a positive constant.

The behavior of solutions as $t \to \infty$ depends crucially on the parameter α. We show in Section 5 that *if* $|\alpha| \leq 1$ *then the rest field is asymptotically stable.* On the other hand, when $\alpha < -1$ there exist oscillatory solutions which *do not* approach zero when $t \to \infty$, while if $\alpha > 1$ there exist solutions which approach *non zero* functions $\psi = \psi(x)$ as $t \to \infty$.

A similar situation occurs when there are higher order dissipation terms present in the equation, for example in the case

$$u_{tt} - \Delta\bigl(u + b(t)\, t^\beta u_t\bigr) + a(t)\, t^\alpha u_t + V(x)u = 0, \tag{1.7}$$

where a, V, and α are as before, and

$$1/D \leq b(t) \leq D \qquad \text{on } J,$$

with $D > 0$ constant. In Section 7 we prove the following result:

The rest field is asymptotically stable for equation (1.7) *provided that*

$$\alpha \le 1, \quad |\beta| \le 1 \qquad \text{or} \qquad \beta \le 1, \quad |\alpha| \le 1. \tag{1.8}$$

Moreover, when $\alpha < 1$ and $\beta < 1$ there exist oscillatory solutions which do not approach zero as $t \to \infty$, while if either $\alpha > 1$ or $\beta > 1$ there are again solutions which approach non zero functions $\psi = \psi(x)$ as $t \to \infty$.

2. Definition of solutions

It is axiomatic that the notion of a classical solution of (1.1) is inadequate not only for showing the existence of solutions, but also for maintaining a reasonable generality of treatment. On the other hand, in research primarily devoted to the existence of solutions of (1.1) it has been necessary to assume special conditions on the functions f and Q in order for the existence process to be carried out successfully. This leaves open whether one can frame a definition of solution which is suitably general, for existence purposes independent of detailed properties of f and Q, and adequate for the investigation of asymptotic stability.

To provide such a definition it is convenient first to introduce the elementary bracket pairing in $\Omega \subset \mathbb{R}^n$,

$$\langle \varphi, \psi \rangle \equiv \int_\Omega (\varphi, \psi) dx,$$

provided that $(\varphi, \psi) \in L^1(\Omega)$. We write for simplicity

$$L^p = \left[L^p(\Omega)\right]^N, \qquad X = \left[W_0^{1,2}(\Omega)\right]^N,$$

where $p > 1$, these spaces being endowed respectively with the natural norms

$$\|\varphi\|_{L^p} = \left(\int_\Omega |\varphi|^p dx\right)^{1/p}, \qquad \|D\varphi\|^2 = \int_\Omega |D\varphi|^2 dx = \int_\Omega \sum_{i=1}^n |D_i\varphi|^2 dx.$$

When $p = 2$ it is convenient to write simply $\|\varphi\|$ instead of $\|\varphi\|_{L^2}$. We also put

$$\langle D\varphi, D\psi \rangle = \int_\Omega \sum_{i=1}^n (D_i\varphi, D_i\psi) dx \qquad \text{for all } \varphi, \psi \in X,$$

so in particular $\langle D\varphi, D\varphi \rangle = \|D\varphi\|^2$.

Now define $K' = C(J \to X) \cap C^1(J \to L^2)$ and

$$K = \{\phi \in K' \ : \ E\phi \text{ is locally bounded on } J\},$$

where $E\phi$ is the *total energy of the field* ϕ, that is

$$E\phi = E\phi(t) = \tfrac{1}{2}\|\phi'\|^2 + \tfrac{1}{2}\|D\phi\|^2 + \mathcal{F}\phi,$$

and $\mathcal{F}\phi$, the *potential energy of the field*, is given by

$$\mathcal{F}\phi = \mathcal{F}\phi(t) = \int_\Omega F(x, \phi(t, x))dx.$$

The quantity ϕ' in $E\phi$ can be written more explicitly as $\phi'(t)$, where $\phi'(t) = (d\phi/dt)(t) \in L^2$ for each $t \in J$. Similarly, again since $\phi \in K$, we have $D\phi = D\phi(t) \in L^2$ for each $t \in J$. Of course, $\|\phi'\|$ and $\|D\phi\|$ are then continuous functions of t.

In writing $E\phi$ and $\mathcal{F}\phi$ we make the tacit agreement that $\mathcal{F}\phi$ is *well–defined*, namely that $F(\cdot, \phi(t, \cdot)) \in L^1$ for all $t \in J$. An equivalent definition for K is clearly given by

$$K = \{\phi \in K' \ : \ \mathcal{F}\phi \text{ is locally bounded on } J\}.$$

Our motivation for introducing the set K is that solutions of (1.1) should, whatever else, be sought in a function space for which the total energy is well–defined and bounded on any finite time interval, and K has essentially just this property.

The definition of K moreover applies without reference to the restoring force condition (1.3). Indeed, *throughout this section we shall not make use of* (1.3), so that the definition of solution given below applies equally whether or not f represents a restoring force. Of course f must still be derivable from a potential as in (1.4).

We can now give our principal definition: a *strong solution of* (1.1) is a function $u \in K$ satisfying the following *two conditions*:

(A) *Distribution Identity*

$$\langle u', \phi\rangle\Big]_0^t = \int_0^t \{\langle u', \phi'\rangle - \langle Du, D\phi\rangle - \langle Q(s, \cdot, u, u'), \phi\rangle - \langle f(\cdot, u), \phi\rangle\}ds$$

for all $t \in J$ and $\phi \in K$.

(B) *Conservation Law*

(i) $$\langle Q(t, \cdot, u, u'), u'\rangle \in L^1_{\mathrm{loc}}(J),$$

(ii) $$Eu\Big]_0^t = -\int_0^t \langle Q(s, \cdot, u, u'), u'\rangle ds \quad \text{for all } t \in J.$$

We emphasize that *condition* (B) *is an essential attribute of solutions.* Indeed, standard existence theorems for (1.1) in the literature always yield solutions satisfying *both*

(A) and (B). On the other hand, (A) alone does not imply (B), even if the integrability condition (B-i) is assumed a priori, see [16].

A remaining, crucial issue is to determine a category of functions f and Q for which the preceding definition is meaningful. In particular, it must be shown that

$$\langle f(\cdot,u),\phi(t,\cdot)\rangle,\ \langle Q(t,\cdot,u,u'),\phi(t,\cdot)\rangle \in L^1_{\text{loc}}(J) \tag{2.1}$$

so that the right hand integral in the identity (A) will be well–defined. To obtain (2.1), observe first that if u, $\phi\in K$, then

$$u,\ \phi\in C(J\,\to\,L^r), \tag{2.2}$$

where $r=2n/(n-2)$ is the Sobolev exponent for $\mathbb{R}^n$ ($2<r<\infty$ if $n=1,2$, since Ω is bounded).

We make the following natural hypotheses on f and Q, in the principal case $n\geq 2$:

(H1) There exists an exponent $p>1$ such that

$$|f(x,u)|\leq \text{Const.}\,[1+|u|^{p-1}].$$

Moreover, if $n\geq 3$ and $p>r$, there are constants $\kappa>0$ and $\kappa_1\geq 0$ for which

$$\big(f(x,u),u\big)\geq \kappa|u|^p-\kappa_1|u|.$$

$$|Q(t,x,u,v)|\leq \delta(t)\,[1+|u|^{q-1}+|v|^{q-1}], \tag{H2}$$

where $\delta\in L^1_{\text{loc}}(J\,\to\,\mathbb{R}^+_0)$ and

$$1<q\leq\mu\quad (1<q<\infty \text{ if } n=2);\qquad \mu=\max\{p,\ r\}.$$

(H3) Q is *tame*, that is, there exists $\gamma\geq 1$ such that

$$|Q(t,x,u,v)|\cdot|v|\leq\gamma\big(Q(t,x,u,v),v\big)\qquad (\text{automatic if } N=1).$$

Note that each of these conditions is assumed to hold for all values of the given arguments, namely, $t\in J$, $x\in\Omega$, $u\in\mathbb{R}^N$, $v\in\mathbb{R}^N$. The case $n=1$ is treated separately at the end of the section.

It is worth observing that the homogeneous function $f(u)=|u|^{p-2}u$ satisfies (H1), while the function $Q(v)=|v|^{q-2}v$ satisfies (H2) and (H3), provided that $1<q\leq\mu$.

The following lemmas will be used to show that (2.1) is satisfied.

Lemma 2.1. *Let* (H1) *hold. If* $n \geq 3$, *then* $\phi(t,\cdot) \in L^\mu$ *for all* $\phi \in K$ *and* $t \in J$, *and*

$$\|\phi\|_{L^\mu} \quad \text{is locally bounded on } J. \tag{2.3}$$

If $n = 2$, *then* $\phi(t,\cdot) \in L^q$ *for all* $\phi \in K$ *and* $q > 1$; *moreover* $\|\phi\|_{L^q}$ *locally bounded on* J.

Proof: In view of (2.2) we need only consider the case $n \geq 3$, $p > r$. Obviously

$$F(x,u) = \int_0^1 \big(f(x,tu),u\big)dt, \qquad u \in \mathbb{R}^N, \tag{2.4}$$

so by the second condition of (H1)

$$F(x,u) \geq \int_0^1 (\kappa\,|u|^p t^{p-1} - \kappa_1|u|)dt \geq \frac{\kappa}{p}\,|u|^p - \kappa_1|u|.$$

Consequently, for $\phi \in K$ we have

$$\|\phi\|_{L^p}^p \leq \frac{p}{\kappa}\,\mathcal{F}\phi + \frac{p\kappa_1}{\kappa}\|\phi\|_{L^1}, \quad t \in J.$$

This proves the result, since both $F(x,\phi)$ and $|\phi|$ are integrable on Ω, and $\mathcal{F}\phi$ and $\|\phi\|_{L^1}$ are locally bounded on J; see the definition of K.

Lemma 2.2. *Let* (H1) *hold. Then*

$$|\mathcal{F}\phi| \leq \text{Const.}\big(\|\phi\|_{L^1} + \|\phi\|_{L^p}^p\big) \tag{2.5}$$

for all $\phi \in K$, *and*

$$|\langle f,\phi\rangle| \leq \text{Const.}\big(\|\phi\|_{L^1} + \|u\|_{L^p}^{p-1}\cdot\|\phi\|_{L^p}\big) \qquad \text{for } u,\phi \in K. \tag{2.6}$$

Proof: From (2.4) and the first part of (H1) we have

$$|F(x,u)| \leq \text{Const.}\,(|u| + |u|^p), \qquad u \in \mathbb{R}^N, \tag{2.7}$$

and (2.5) follows at once. Condition (2.6) is equally clear.

Lemma 2.1 and (2.6) show that $\langle f(\cdot,u),\phi\rangle$ is locally bounded on J whenever u, $\phi \in K$, so that this term in (A) is well defined.

Lemma 2.3. *Let $u \in K$ satisfy* (B-i). *Suppose that* (H2) *and* (H3) *hold. Then for all $\phi \in K$ and $t \in J$*

$$\int_0^t |\langle Q(s,\cdot,u,u'),\phi(s,\cdot)\rangle| ds$$
$$\leq \left\{ c + 2\gamma \left(\int_0^t \langle Q(s,\cdot,u,u'),u'\rangle ds \right)^{1/q'} \right\} \cdot \left(\int_0^t \delta(s) ds \right)^{1/q} \cdot \sup_{[0,t]} \|\phi(s,\cdot)\|_{L^q},$$

where c is a constant depending only on q, $|\Omega|$, $\|\delta\|_{L^1(0,t)}$, $\sup_{[0,t]} \|u(s,\cdot)\|_{L^q}$.

Proof: Write

$$\langle Q(s,\cdot,u,u'),\phi\rangle = \left(\int_{\Omega_1} + \int_{\Omega_2} \right) (Q,\phi) dx, \qquad s \in J, \tag{2.8}$$

where

$$\Omega_1 = \Omega_1(s) = \{x \in \Omega \ : \ |u'(s)(x)|^{q-1} \geq 1 + |u(s)(x)|^{q-1}\},$$
$$\Omega_2 = \Omega_2(s) = \{x \in \Omega \ : \ |u'(s)(x)|^{q-1} \leq 1 + |u(s)(x)|^{q-1}\};$$

here $u(s)(x)$ denotes the value of the function $u(s,\cdot) \in L^q$ at the point $x \in \Omega$, and similarly for $u'(s)(x)$. By Hőlder's inequality and (H2)–(H3), we get

$$\begin{aligned} \left| \int_{\Omega_1} (Q,\phi) dx \right| &\leq \left(\int_{\Omega_1} |Q| \cdot |u'| dx \right)^{1/q'} \left(\int_{\Omega_1} |Q| \cdot |u'|^{1-q} \cdot |\phi|^q dx \right)^{1/q} \\ &\leq \left(\gamma \int_{\Omega} (Q,u') dx \right)^{1/q'} \cdot [2\delta(s)]^{1/q} \cdot \|\phi\|_{L^q}. \end{aligned} \tag{2.9}$$

Also, by (H2),

$$\begin{aligned} \left| \int_{\Omega_2} (Q,\phi) dx \right| &\leq \left(\int_{\Omega_2} |Q|^{q'} dx \right)^{1/q'} \left(\int_{\Omega_2} |\phi|^q dx \right)^{1/q} \\ &\leq 4\delta(s) \cdot \left(\int_{\Omega} (1 + |u|^q) dx \right)^{1/q'} \cdot \|\phi\|_{L^q}. \end{aligned} \tag{2.10}$$

The lemma now follows by integration of (2.8) over $(0,t)$, and another use of Hőlder's inequality to estimate the time integral of (2.9).

Lemma 2.1 and the boundedness of Ω show that $\|\phi\|_{L^q}$ is locally bounded on J, see (H2). Hence Lemma 2.3 implies that $\langle Q(t,\cdot,u,u'),\phi\rangle \in L^1_{\text{loc}}(J)$ when u, $\phi \in K$. In turn this term in (A) is also well defined, as required.

The next lemma gives an alternative definition for the set K, when (H1) holds. While it is not required in the sequel, it is worth stating in order to clarify further the meaning of K.

LEMMA 2.4. *Let* (H1) *be satisfied. Then, for* $n \geq 3$,

$$K \equiv \{\phi \in K' \ : \ \phi(t,\cdot) \in L^\mu \ \textit{for all } t \in J, \textit{ and } \|\phi\|_{L^\mu} \textit{ is locally bounded on } J.\}$$

Proof: Let $\hat{K}$ denote the set on the right side above. By Lemma 2.1, clearly $K \subset \hat{K}$. On the other hand, if $\phi \in \hat{K}$ then by the argument of Lemma 2.2, see (2.7), we get $F(\cdot, \phi(t,\cdot)) \in L^1$ for all $t \in J$, so that $\mathcal{F}\phi$ is well–defined on J. In turn, using (2.5) and the fact that $p \leq \mu$, we infer that $\mathcal{F}\phi$ is locally bounded on J.

COROLLARY. *When* $n = 2$*, or when* $p \leq r$ *in* (H1)*, we have* $K \equiv K'$.

LEMMA 2.5. *If* (1.3) *holds then* $\mathcal{F}\phi \geq 0$ *and* $E\phi \geq 0$ *for all* $\phi \in K$ *and* $t \in J$.

This follows at once from (2.4).

Remarks. Concerning the case $n \geq 3$, $p > r$ in hypothesis (H1), it is important to observe that a general existence theorem for such rapidly growing potentials is somewhat unlikely. At the same time, there do exist special solutions in such cases, e.g. the equation

$$u_{tt} - \Delta u + (1 + \mu + |u_t|^{p-2})u_t + |u|^{p-2}u = 0,$$

where μ is an eigenvalue of the Laplacian, has the explicit solution $e^{-t}\varphi(x)$, provided that φ is an eigenfunction corresponding to μ. Thus results for $p > r$ still remain of interest for the problem of asymptotic stability, justifying the relatively small additional effort involved in treating this situation.

A final comment concerns the terminology *strong solution* in the definition (A), (B). Our purpose is to emphasize the fact that in (A) the test space for the functions ϕ is precisely the set K in which solutions reside, while in contrast the typical test space for distribution (*weak*) solutions is the smaller set $C_c^\infty(J \times \Omega)$. In general it seems not possible to obtain stability results for solutions of the latter type, a fact which motivates our choice of the test space K and the distinction made between weak and strong solutions.

In the same regard, note that, as before, conditions (H1)–(H3) are still needed for condition (A) to be meaningful in the case of weak solutions, with the exception that it is no longer necessary to have $q \leq \mu$.

The case $n = 1$.

When $n = 1$ we can again take $K \equiv K'$. Moreover, the hypothesis (H1) can be weakened to the form

(H1)′ $$|f(x,u)| \leq g(u), \qquad g \in C(\mathbb{R}^N \to \mathbb{R}_0^+),$$

for all $(x,u) \in \Omega \times \mathbb{R}^N$.

Lemma 2.2′. *Let* (H1)′ *hold. Then* $\mathcal{F}\phi$ *is well–defined and locally bounded on* J *for all* $\phi \in K$, *while also*

$$|\langle f(\cdot,u),\phi(t,\cdot)\rangle| \le \psi(t)\|\phi(t,\cdot)\|_{L^1}, \qquad t\in J,$$

for all u, $\phi\in K$, *where* $\psi \in L^\infty_{\mathrm{loc}}(J)$.

Proof: Because $n=1$ and $\phi(t)\in X$ for all $t\in J$, we get

$$\|\phi(t)\|_{L^\infty} \le \sqrt{|\Omega|/2}\;\|D\phi(t)\| \in C(J).$$

Thus, as in the proof of Lemma 2.2, the conclusions follow from (H1)′.

Remark. Condition (H1)′ is automatic if $f\in C(\overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N)$, or if f does not depend on x.

Hypothesis (H2) can similarly be weakened to the form

$$\text{(H2)}' \qquad |Q(t,x,u,v)| \le \delta(t)g(u)\,[1+|v|^{q-1}],$$

where

$$\delta\in L^1_{\mathrm{loc}}(J\to\mathbb{R}^+_0), \qquad g\in C(\mathbb{R}^N\to\mathbb{R}^+_0).$$

Lemma 2.3′. *Let* $u\in K$ *satisfy* (B-i). *Suppose that* (H2)′ *and* (H3) *hold. Then for all* $\phi\in K$ *and* $t\in J$

$$\int_0^t |\langle Q(s,\cdot,u,u'),\phi(s,\cdot)\rangle|ds$$
$$\le c\left\{1+\left(\int_0^t \langle Q(s,\cdot,u,u'),u'\rangle ds\right)^{1/q'}\right\}\cdot\left(\int_0^t \delta(s)ds\right)^{1/q}\cdot\sup_{[0,t]}\|\phi(s,\cdot)\|_{L^q},$$

where c *is a constant depending only on* q, γ, g, $|\Omega|$, $\|\delta\|_{L^1(0,t)}$, $\sup_{[0,t]}\|u(s,\cdot)\|_{L^\infty}$.

The proof is essentially the same as for Lemma 2.3.

Classical solutions

Although classical solutions of (1.1) are unlikely except in special circumstances, nevertheless it is worth commenting on their treatment within the present context.

A classical solution is a function $u=u(t,x)\in C^2(J\times\overline{\Omega})$ satisfying (1.1) in the natural sense. Let

$$K^* = \{\phi\in C^2(J\times\overline{\Omega}) \;:\; \phi(t,x)=0 \text{ when } x\in\partial\Omega\},$$

and suppose that (H1)$'$, and (H2)$'$ hold. If we multiply (1.1) by $\phi \in K^*$ and integrate by parts in both t and x, then we get the distribution identity (A), while if we multiply by u_t then similarly (B) is obtained.

Thus, if u is a classical solution then $u \in K^*$, while (A) holds with $\phi \in K^*$ and (B) is satisfied. The previous results of course continue to hold, as in the case $n = 1$.

3. Asymptotic Stability, Part I

In this and the following section we give our main results on the global asymptotic stability of solutions of (1.1). We recall that a solution of (1.1) is a function $u \in K$ satisfying the requirements (A) and (B) in the previous section.

We retain the principal conditions (1.2), (1.3), (1.4) together with hypotheses (H1), (H3). In addition, the following stronger version of (H2) will be required:

$$\text{(AS)} \qquad \sigma(t)\omega(|v|) \le |Q(t,x,u,v)| \le \delta(t)\,[|v|^{m-1} + |v|^{q-1}],$$

where

$$1 < m \le q, \qquad 1 < q \le \mu \quad (1 < q < \infty \text{ if } n = 2),$$

$$\sigma^{1-m},\ \delta \in L^1_{\mathrm{loc}}(J \to \mathbb{R}^+),$$

and $\omega \in C(\mathbb{R}^+_0 \to \mathbb{R}^+_0)$ is such that

$$\omega(0) = 0; \qquad \omega(s) > 0 \quad \text{for } 0 < s < 1, \qquad \omega(s) = s \quad \text{for } s \ge 1.$$

Here it is tacitly assumed that $n \ge 2$; the case $n = 1$ is treated briefly at the end of the section.

THEOREM 3.1. *Let* (H1)*,* (H3) *and* (AS) *hold. Suppose there exists* $k \in CBV(J \to \mathbb{R}^+_0)$ *such that*

$$k \notin L^1(J), \tag{3.1}$$

$$\liminf_{t\to\infty} \int_0^t (\delta + \sigma^{1-m}) k^m ds \Bigg/ \left(\int_0^t k ds\right)^m < \infty. \tag{3.2}$$

Then along any strong solution u *of* (1.1) *we have*

$$\lim_{t\to\infty} Eu(t) = 0. \tag{3.3}$$

The integral condition (3.2) prevents the damping term Q being either too small (*underdamping*), or too large (*overdamping*) as $t \to \infty$. In Section 5 we present a number of corollaries which make this remark more explicit.

The proof of Theorem 3.1 is based on several preliminary lemmas, using arguments related to those in [14].

In what follows $u = u(t) \in K$ denotes a given solution of (1.1) in J, in the sense of Section 2.

Lemma 3.1. *The total energy Eu is a Lyapunov function, that is,*

$$Eu \geq 0 \qquad \text{and} \qquad Eu \quad \text{is decreasing on } J. \tag{3.4}$$

Moreover,

$$\|u'\|,\ \|Du\| \in L^\infty(J), \qquad \|u\|,\ \|u\|_{L^r},\ \|u\|_{L^p} \in L^\infty(J), \tag{3.5}$$

and

$$\langle Q(t,\cdot,u,u'),u'\rangle \in L^1(J). \tag{3.6}$$

Proof: Condition $(3.4)_1$ follows from Lemma 2.5, and $(3.4)_2$ from (B) and (1.2). In turn, $Eu \in L^\infty(J)$. Consequently, each of the terms $\|u'\|$, $\|Du\|$ and $\mathcal{F}u$ making up Eu must be bounded, since each is non–negative. Then, from the argument of Lemma 2.1, we find that $\|u\|_{L^p}$ is bounded, and from Sobolev, that $\|u\|_{L^r}$ is bounded. Finally $\|u\|$ is bounded as a consequence of Hőlder's inequality and the boundedness of Ω.

Again using (1.2) and the fact that Eu is bounded on J, it is clear from (B-ii) that (3.6) is valid.

Lemma 3.2. *Suppose that* (AS) *and* (H3) *hold. Then, for any $T \in J$ and $t \geq T$, we have*

$$\int_T^t k\,|\langle Q(s,\cdot,u,u'),u\rangle| ds \leq \varepsilon(T)\left[1+\left(\int_0^t \delta\, k^m ds\right)^{1/m}\right], \tag{3.7}$$

where $\varepsilon(T) \to 0$ as $T \to \infty$.

Proof: Define, for any fixed $s \in J$,

$$\Omega_1 = \Omega_1(s) = \{x \in \Omega \ :\ |u'(s)(x)| \leq 1\},$$
$$\Omega_2 = \Omega_2(s) = \{x \in \Omega \ :\ |u'(s)(x)| \geq 1\}.$$

Then, as in the demonstration of Lemma 2.3, we get by (AS) and (H3),

$$k\left|\int_{\Omega_1}\big(Q(s,\cdot,u,u'),u\big)dx\right| \leq (2\delta\, k^m)^{1/m}\cdot\left(\gamma\int_\Omega \big(Q(s,\cdot,u,u'),u'\big)dx\right)^{1/m'}\|u(s)\|_{L^m}$$

and

$$k\left|\int_{\Omega_2}\big(Q(s,\cdot,u,u'),u\big)dx\right| \leq (2\delta\, k^q)^{1/q}\cdot\left(\gamma\int_\Omega \big(Q(s,\cdot,u,u'),u'\big)dx\right)^{1/q'}\|u(s)\|_{L^q}.$$

Consequently, integrating with respect to s from T to t, we find, by Hőlder's inequality,

$$\int_T^t k\,|\langle Q(s,\cdot,u,u'),u\rangle|ds$$

$$\leq 2\gamma\left(\int_T^t \delta\, k^m ds\right)^{1/m}\left(\int_T^t \langle Q(s,\cdot,u,u'),u'\rangle ds\right)^{1/m'}\sup_{[T,t]}\|u(s)\|_{L^m}$$

$$+\,2\gamma\left(\int_T^t \delta\, k^q ds\right)^{1/q}\left(\int_T^t \langle Q(s,\cdot,u,u'),u'\rangle ds\right)^{1/q'}\sup_{[T,t]}\|u(s)\|_{L^q}.$$

Since k is of bounded variation on J, we have $k(t) \leq \text{Const.} = k_0$ on J. Moreover $m \leq q \leq \max\{p,\ r\}$, so that by Hőlder's inequality and (3.5)

$$\|u(s)\|_{L^m} \leq \text{Const.}\,\|u(s)\|_{L^q} \leq \text{Const.},$$

$$\left(\int_T^t \delta\, k^q ds\right)^{1/q} \leq k_0^{(q-m)/q}\left[1+\left(\int_T^t \delta\, k^m ds\right)^{1/m}\right].$$

Thus,

$$\int_T^t k\,|\langle Q(s,\cdot,u,u'),u\rangle|ds \leq \text{Const.}\left[1+\left(\int_T^t \delta\, k^m ds\right)^{1/m}\right]$$

$$\cdot\left\{\left(\int_T^t \langle Q(s,\cdot,u,u'),u'\rangle ds\right)^{1/q'}+\left(\int_T^t \langle Q(s,\cdot,u,u'),u'\rangle ds\right)^{1/m'}\right\},$$

where the constant depends only on q, m, k_0, γ, $|\Omega|$, and $\sup_J \|u(s)\|_{L^q}$. The proof is now complete, since $\langle Q(s,\cdot,u,u'),u'\rangle \in L^1(J)$ by Lemma 3.1.

Lemma 3.3. *Suppose that* (AS) *and* (H3) *hold. Let ϑ be a given positive constant. Then there exists $C(\vartheta) > 0$ such that for all $t \geq T \geq 0$*

$$\int_T^t k\|u'\|^2 ds \leq \vartheta\int_T^t k ds + \varepsilon(T)C(\vartheta)\left(\int_T^t \sigma^{1-m}k^m ds\right)^{1/m}, \tag{3.8}$$

where $\varepsilon(T) \to 0$ as $T \to \infty$.

Proof: For fixed $s \in J$, define

$$\Omega_1 = \Omega_1(s) = \{x \in \Omega\ :\ |u'(s)(x)| \leq \sqrt{\vartheta/|\Omega|}\},$$
$$\Omega_2 = \Omega_2(s) = \{x \in \Omega\ :\ |u'(s)(x)| \geq \sqrt{\vartheta/|\Omega|}\}.$$

Clearly

$$\int_{\Omega_1} |u'|^2 dx \le \vartheta, \tag{3.9}$$

while, by Hőlder's inequality,

$$\int_{\Omega_2} |u'|^2 dx \le \int_{\Omega_2} \frac{|u'|}{\omega(|u'|)} \cdot \omega(|u'|)\,|u'| dx, \tag{3.10}$$

where ω is the function appearing on the left hand side of (AS). Now

$$\sup_{\tau \ge \sqrt{\vartheta/|\Omega|}} \frac{\tau}{\omega(\tau)} = \sup_{\sqrt{\vartheta/|\Omega|} \le \tau \le 1} \frac{\tau}{\omega(\tau)} = \Lambda, \qquad \Lambda = \Lambda(\vartheta) \ge 1, \tag{3.11}$$

since ω is continuous and positive for $\tau > 0$, and $\omega(\tau) = \tau$ for $\tau \ge 1$. From (3.10), (3.11) and $(\text{AS})_1$ we obtain

$$\begin{aligned} \int_{\Omega_2} |u'|^2 dx &\le \Lambda(\vartheta) \int_{\Omega_2} \frac{|Q(s,\cdot,u,u')| \cdot |u'|}{\sigma(s)} dx \\ &\le \frac{\gamma \Lambda(\vartheta)}{\sigma(s)} \int_{\Omega} \big(Q(s,\cdot,u,u'),u'\big) dx. \end{aligned}$$

In turn, since $\|u'\| \le$Const. in J,

$$\begin{aligned} k\int_{\Omega_2} |u'|^2 dx &\le \text{Const.}\, k \left(\int_{\Omega_2} |u'|^2 dx \right)^{1/m'} \\ &\le \text{Const.}\, [\gamma\Lambda(\vartheta)]^{1/m'} (\sigma^{1-m} k^m)^{1/m} \big(\langle Q(s,\cdot,u,u' \rangle \big)^{1/m'}, \end{aligned}$$

so that (3.9) yields

$$k\|u'(s)\|^2 \le \vartheta k + \text{Const.}\, [\gamma\Lambda(\vartheta)]^{1/m'} (\sigma^{1-m} k^m)^{1/m} \big(\langle Q(s,\cdot,u,u'),u' \rangle \big)^{1/m'}.$$

The required result now follows by integration from T to t and another use of Hőlder's inequality; in particular, we can take $C(\vartheta) = [\Lambda(\vartheta)]^{1/m'}$, and

$$\varepsilon(T) = \text{Const.} \left(\int_T^\infty \langle Q(s,\cdot,u,u'),u' \rangle ds \right)^{1/m'} \to 0 \qquad \text{as } T \to \infty$$

by (3.6).

Note that $C(\vartheta) \to \infty$ as $\vartheta \to 0$, whenever $\omega(\tau) = o(\tau)$ as $\tau \to 0$, as can be seen from the calculation (3.11).

By (3.4) it is clear that there exists $\ell \ge 0$ such that

$$\lim_{t\to\infty} Eu(t) = \ell. \tag{3.12}$$

LEMMA 3.4. *Suppose* $\ell > 0$ *and that* (H1) *holds. Then there exists* $\alpha = \alpha(\ell) > 0$ *such that*

$$\|u'\|^2 + \|Du\|^2 + \langle f(\cdot, u), u\rangle \geq \alpha \qquad \text{on } J. \tag{3.13}$$

Proof: Since $Eu(t) \geq \ell$ for all $t \in J$, it follows that

$$\|u'\|^2 + \|Du\|^2 \geq 2(\ell - \mathcal{F}u) \qquad \text{on } J.$$

Let

$$J_1 = \{t \in J \ :\ \mathcal{F}u(t) \leq \ell/2\}, \qquad J_2 = \{t \in J \ :\ \mathcal{F}u(t) > \ell/2\}.$$

For $t \in J_1$

$$\|u'(t)\|^2 + \|Du(t)\|^2 \geq \ell.$$

Next, consider $t \in J_2$. There are two cases, depending on whether $p \leq r$ or $p > r$. In the first case we obtain by (2.5)

$$\tfrac{1}{2}\ell < \mathcal{F}u(t) \leq C_1(\|u(t)\|_{L^1} + \|u(t)\|_{L^p}^p). \tag{3.14}$$

Using the Sobolev theorem,

$$\|u(t)\|_{L^r} \leq C\,\|Du(t)\|, \tag{3.15}$$

and in turn

$$\begin{aligned} \|u(t)\|_{L^1} &\leq |\Omega|^{1/r'}\|u(t)\|_{L^r} \leq C\,|\Omega|^{1/r'}\|Du(t)\| \\ &\leq \varepsilon + C^p|\Omega|^{p/r'}\varepsilon^{1-p}\|Du(t)\|^p \end{aligned} \tag{3.16}$$

by Young's inequality, for any $\varepsilon > 0$.

Hence by (3.14)–(3.16)

$$\tfrac{1}{2}\ell \leq C_1\varepsilon + C_1C^p|\Omega|^{1-p/r}[1 + (|\Omega|/\varepsilon)^{p-1}] \cdot \|Du(t)\|^p.$$

Choosing $\varepsilon = \ell/4C_1$ then gives

$$\|Du(t)\|^p \geq \frac{\ell}{4C_1C^p|\Omega|^{1-p/r}[1 + (4C_1|\Omega|/\ell)^{p-1}]} = C_2(\ell), \qquad t \in J_2.$$

Since $\langle f, u\rangle \geq 0$ this proves (3.13) for the first case, with

$$\alpha = \min\{\ell,\ C_2^{2/p}(\ell)\}.$$

Next, if $p > r$, we have by Lemma 2.2

$$\begin{aligned}\tfrac{1}{2}\ell < \mathcal{F}u(t) &\le C_1(\|u(t)\|_{L^1} + \|u(t)\|^p_{L^p}) \\ &\le C_1\left\{\|u(t)\|_{L^1} + \frac{1}{\kappa}\big(\langle f,u\rangle + \kappa_1\|u(t)\|_{L^1}\big)\right\},\end{aligned}$$

where the last step follows from the second part of (H1). But, as in (3.16),

$$\|u(t)\|_{L^1} \le \varepsilon + C^2\,\frac{|\Omega|^{2/r'}}{\varepsilon}\,\|Du\|^2,$$

which yields

$$\tfrac{1}{2}\ell < \frac{C_1}{\kappa}\,\langle f,u\rangle + C_1C^2\left(1+\frac{\kappa_1}{\kappa}\right)\frac{|\Omega|^{2/r'}}{\varepsilon}\|Du\|^2 + C_1\left(1+\frac{\kappa_1}{\kappa}\right)\varepsilon.$$

Choosing $\varepsilon = \ell\kappa/4C_1(\kappa+\kappa_1)$, then gives

$$\frac{C_3^2}{4\ell}\,\|Du\|^2 + \frac{C_1}{\kappa}\langle f,u\rangle \ge \frac{\ell}{4}, \qquad t\in J_2,$$

where

$$C_3 = 4\,C\,C_1\left(1+\frac{\kappa_1}{\kappa}\right)^2|\Omega|^{1/r'}.$$

Since both $\|Du\|^2$ and $\langle f,u\rangle$ are non–negative, this again implies (3.13), with

$$\alpha = \min\left\{\ell,\ \frac{\kappa\ell}{4C_1},\ \left(\frac{\ell}{C_3}\right)^2\right\}.$$

This completes the proof.

Proof of Theorem 3.1: First we treat the simpler case in which k is not only $CBV(J)$, but also of class $C^1(J)$. Suppose for contradiction that $\ell > 0$ in (3.12). Define a second Lyapunov function by

$$V(t) = k(t)\langle u,u'\rangle = \langle u',\phi\rangle, \qquad \phi = k(t)u. \tag{3.17}$$

Since $k \in C^1(J)$, it is clear that $\phi \in K$. Hence, by the distribution identity (A) in Section 2, we have for any $t \ge T \ge 0$

$$\begin{aligned}V(s)\big]_T^t = \int_T^t &\{k'\langle u,u'\rangle + 2k\|u'\|^2 \\ &- k(\|u'\|^2 + \|Du\|^2 + \langle f(\cdot,u),u\rangle) - k\langle Q(s,\cdot,u,u'),u\rangle\}ds.\end{aligned} \tag{3.18}$$

Note that an additional term $k\|u'\|^2$ has been added and subtracted inside the integral. (It is exactly here that we use the fact that u is a *strong* solution of (1.1); otherwise, if u is only a weak solution, then $\phi = k\,u$ need not be in the test space $C_c^\infty(J \times \Omega)$ and (3.18) can no longer be inferred.)

We now estimate the right hand side of (3.18). First

$$|\langle u, u' \rangle| \le \|u\| \cdot \|u'\| \le \text{Const.} \qquad \text{in } J$$

by (3.5). Applying Lemmas 3.2–3.4 to the remaining terms, we then obtain

$$V(s)\big]_T^t \le \text{Const.} \int_T^\infty |k'| ds + 2\vartheta \int_T^t k ds + 2\varepsilon(T) C(\vartheta) \left(\int_0^t \sigma^{1-m} k^m ds \right)^{1/m} - \alpha \int_T^t k ds + \varepsilon(T) \left\{ 1 + \left(\int_0^t \delta k^m ds \right)^{1/m} \right\}.$$

Choose $\vartheta = \vartheta(\ell) = \alpha/4$ and fix $T > 0$ sufficiently large so that

$$\text{Const.} \int_T^\infty |k'| ds \le \tfrac{1}{2}, \qquad \varepsilon(T) \le \tfrac{1}{2}.$$

Then

$$V(s)\big]_T^t \le 1 - \frac{\alpha}{2} \int_T^t k ds + \varepsilon(T) \left\{ 2\,C(\vartheta) \left(\int_0^t \sigma^{1-m} k^m ds \right)^{1/m} + \left(\int_0^t \delta k^m ds \right)^{1/m} \right\}.$$

By (3.2) there is a sequence $t_i \nearrow \infty$ and a number $M > 0$ such that

$$\int_0^{t_i} (\delta + \sigma^{1-m}) k^m ds \le \left(M \int_0^{t_i} k ds \right)^m.$$

Consequently, for $t_i \ge T$,

$$V(t_i) - V(T) \le 1 - \frac{\alpha}{2} \int_T^{t_i} k ds + \varepsilon(T) \left[2\,C(\vartheta) + 1 \right] M \int_T^{t_i} k ds. \tag{3.19}$$

We now take T even larger, if necessary, so that

$$\varepsilon(T)[2\,C(\vartheta) + 1] M \le \alpha/4,$$

which gives, for all $t_i \ge T$,

$$V(t_i) \le V(T) + 1 - \frac{\alpha}{4} \int_T^{t_i} k ds.$$

Thus by (3.1) we get

$$\lim_{i\to\infty} V(t_i) = -\infty.$$

On the other hand, recalling that k is bounded,

$$|V(t)| \le \sup_J k \cdot \|u(t)\| \cdot \|u'(t)\| \le \text{Const.} \qquad \text{for } t \in J$$

by (3.5). This contradiction completes the first part of the proof.

For the general case, in which $k \notin C^1(J)$, the proof is based on Lemma A [14, p. 290]. We take $\theta = 2$ in this lemma and let $\overline{k} \in C^1(J)$ and $E \subset J$ be an open set as there, namely such that

$$\text{(i)} \quad 2k \ge \overline{k} \ge \begin{cases} k & \text{in } J \setminus E; \\ 0 & \text{in } E \end{cases}; \qquad \text{(ii)} \quad \operatorname{Var} \overline{k} \le 2 \operatorname{Var} k;$$

$$\text{(iii)} \quad \int_E k ds \le 1.$$

Clearly, $\overline{k} \in CBV(J)$ by (ii).

We assert that $\overline{k}$ satisfies (3.1) and (3.2). Indeed, let T_1 be such that

$$\int_0^{T_1} k ds \ge 2,$$

and consider values $t \ge T_1$. Then, by (i) and (iii)

$$\int_0^t \overline{k} ds \ge \int_{[0,t]\setminus E} k ds \ge \int_0^t k ds - \int_E k ds \ge \int_0^t k ds - 1 \ge \tfrac{1}{2} \int_0^t k ds. \tag{3.20}$$

Hence $\overline{k}$ satisfies (3.1). Moreover, by (i) and (3.20),

$$\frac{\int_0^t (\delta + \sigma^{1-m}) \overline{k}^m ds}{\left(\int_0^t \overline{k} ds\right)^m} \le 4^m \frac{\int_0^t (\delta + \sigma^{1-m}) k^m ds}{\left(\int_0^t k ds\right)^m}, \qquad t \ge T_1,$$

which shows that $\overline{k}$ also satisfies (3.2).

The general case is therefore reduced to the situation when k is smooth, and the proof is complete.

Remark. As in Section 2, our hypotheses can be weakened in case $n = 1$. Specifically, in this case (H1) can be replaced by (H1)$'$ and (AS) by

$$\sigma(t)\omega(|v|) \le |Q(t,x,u,v)| \le \delta(t) g(u) \left[|v|^{m-1} + |v|^{q-1}\right], \tag{AS$'$}$$

where

$$1 < m \le q, \qquad 1 < q < \infty, \qquad g \in C(\mathbb{R}^N \to \mathbb{R}_0^+),$$

and σ, δ, ω are the same as in (AS).

The proof of asymptotic stability is the same as before, taking into account the fact that

$$\|u(t)\|_{L^\infty} \le \sqrt{|\Omega|/2}\,\|Du(t)\| \le \text{Const.} \qquad \text{in } J,$$

see (3.5).

Similarly, Theorem 3.1 continues to apply in the case of classical solutions. The proof is essentially the same as for the case $n = 1$, once we observe that the space K is replaced by K^* and that $\phi = k\,u$ is an admissible test function in K^*.

4. Asymptotic Stability, Part II

We suppose as before that conditions (H1), (H3) and (AS) are satisfied. At the same time, we introduce the following more specific behavior for the function ω in (AS)

$$\omega(s) = s^\nu, \quad 0 \le s \le 1; \qquad \omega(s) = s, \quad s \ge 1, \tag{4.1}$$

where $\nu \ge m - 1$. Recall also that we have defined $\mu = \max\{p,\ r\}$ when $n \ge 3$.

THEOREM 4.1. *Suppose there exists a bounded absolutely continuous function k on J, such that* (3.1) *and* (3.2) *are satisfied, and also*

$$|k'| \le \text{Const.}\,\sigma^\lambda k^{1-\lambda} \qquad a.e \text{ in } J, \tag{4.2}$$

where

$$0 < \lambda \le \begin{cases} \min\left\{\dfrac{\mu-1}{\mu},\ \dfrac{1}{\nu+1}\right\}, & \text{if } n \ge 3 \\ \dfrac{1}{\nu+1}, & \text{if } n = 2 \end{cases}. \tag{4.3}$$

Then (3.3) *holds for any strong solution u of* (1.1).

Proof: We give only the case $n \ge 3$; when $n = 2$ the idea is the same. Since ν in (4.1) can always be made larger, we can clearly suppose without loss of generality that (4.3) has the specific form

$$\lambda = 1/(\nu+1) \le 1 - 1/\mu. \tag{4.4}$$

The proof of Theorem 3.1 now applies almost word for word, except for the estimation of the term

$$\int_T^t k'\langle u, u'\rangle ds.$$

For any $s \in J$, introduce the sets

$$\Omega_1 = \Omega_1(s) = \{x \in \Omega \ : \ |u'(s)(x)| \le 1\},$$
$$\Omega_2 = \Omega_2(s) = \{x \in \Omega \ : \ |u'(s)(x)| \ge 1\}.$$

Then from (4.2)

$$\begin{aligned}\int_{\Omega_1} k'(u,u')dx &\le \text{Const.} \int_{\Omega_1} \sigma^\lambda k^{1-\lambda}|u|\cdot|u'|dx \\ &\le \text{Const.}\, k^{1-\lambda}\left(\int_{\Omega_1}|u|^{1/(1-\lambda)}dx\right)^{1-\lambda}\left(\int_{\Omega_1}\sigma|u'|^{1/\lambda}dx\right)^{\lambda} \\ &\le \text{Const.}\, |\Omega|^{1-\lambda-1/\mu}k^{1-\lambda}\|u\|_{L^\mu}\left(\int_{\Omega_1}\sigma|u'|^{\nu+1}dx\right)^{\lambda}\end{aligned}$$

where by (4.4) we have $1-\lambda-1/\mu \ge 0$, as required for the application of Hőlder's inequality in the third step. Using (AS), (H3), (4.1) and (3.5), now gives

$$\int_{\Omega_1} k'(u,u')dx \le \text{Const.}\, k^{1-\lambda}\left(\int_\Omega \big(Q(s,\cdot,u,u'),u'\big)dx\right)^{\lambda}. \tag{4.5}$$

Next, since $\mu' < 2$ and $|u'(s)| \ge 1$ in Ω_2,

$$\begin{aligned}\int_{\Omega_2} k'(u,u')dx &\le \text{Const.} \int_{\Omega_2} \sigma^\lambda k^{1-\lambda}|u|\cdot|u'|^{2/\mu'}dx \\ &\le \text{Const.}\left(\int_{\Omega_2}\sigma|u'|^2dx\right)^{\lambda}\left(\int_{\Omega_2}k|u|^\mu dx\right)^{1/\mu} \\ &\qquad\qquad \cdot\left(\int_{\Omega_2}k|u'|^2dx\right)^{1-\lambda-1/\mu} \\ &\le \text{Const.}\, k^{1/\mu}|\langle Q(s,\cdot,u,u'),u'\rangle|^\lambda(k\,\|u'\|^2)^{1-\lambda-1/\mu}\end{aligned} \tag{4.6}$$

again by (AS), (H3) and (3.5).

From (4.5), (4.6), and integration from T to $t \geq T$, we now find

$$\int_T^t k'\langle u, u'\rangle ds \leq \text{Const.} \left\{ \left(\int_T^t k ds\right)^{1-\lambda} + \left(\int_T^t k ds\right)^{1/\mu} \left(\int_T^t k\|u'\|^2 ds\right)^{1-\lambda-1/\mu} \right\}$$

$$\cdot \left(\int_T^t \langle Q(s,\cdot,u,u'),u'\rangle ds\right)^{\lambda}$$

$$\leq \text{Const.} \left\{1 + \int_T^t k ds + \int_T^t k\|u'\|^2 ds\right\} \left(\int_T^t \langle Q(s,\cdot,u,u'),u'\rangle ds\right)^{\lambda},$$

by Young's inequality. Finally, taking T sufficiently large yields the estimate

$$\int_T^t k'\langle u, u'\rangle ds \leq \frac{1}{2} + \frac{\alpha}{8}\int_T^t k ds + \int_T^t k\|u'\|^2 ds,$$

where $\alpha = \alpha(\ell)$ is defined in Lemma 3.4.

In conclusion, as in the proof of Theorem 3.1, we find in place of (3.19),

$$V(t_i) - V(T) \leq 1 - \frac{\alpha}{8}\int_T^{t_i} k ds + \varepsilon(T)[3\,C(\vartheta) + 1]\,M \int_T^{t_i} k ds,$$

and this produces the same contradiction as before.

Remarks. *1.* One can combine Theorems 3.1 and 4.1 by assuming that k has the decomposition

$$k = k_1 + k_2,$$

where $k_1 \in CBV(J)$ and $k_2 \in AC(J) \cap L^\infty(J)$ with

$$|k_2'| \leq \text{Const.}\, \sigma^\lambda k^{1-\lambda} \qquad \text{a.e in } J.$$

2. When $n = 1$, we can replace (H1) by (H1)$'$ and (AS) by (AS)$'$.

3. Conditions (AS) and (H3) need not hold in the entire interval $J = [0,\infty)$, but in fact can be restricted to a measurable *control subset* $I \subset J$, see [14, Section 3]. Theorems 3.1 and 4.1 then need to be revised only in two places. First, (3.1) should be modified to the form

$$(3.1)' \qquad k \notin L^1(J), \qquad k = 0 \quad \text{in } J \setminus I,$$

and second (3.2) should be replaced by

$$(3.2)' \qquad \liminf_{t\to\infty} \int_{[0,t]\cap I} (\delta + \sigma^{1-m})k^m ds \Bigg/ \left(\int_0^t k ds\right)^m < \infty.$$

The proofs are exactly the same, provided we agree to write $\delta\, k = \sigma^{1-m} k = 0$ in $J \setminus I$.

The importance of control subsets in studying asymptotic stability is illustrated in [15], in the context of ordinary differential systems.

5. Applications

With appropriate choices of the auxiliary function k we can obtain a number of useful special cases of Theorems 3.1 and 4.1. It is assumed throughout that (H1), (H3) and (AS) are satisfied.

COROLLARY 5.1. *Suppose*

$$\liminf_{t\to\infty} \frac{1}{t^m} \int_0^t (\delta + \sigma^{1-m}) ds < \infty. \tag{5.1}$$

Then (3.3) *holds for any strong solution* u *of* (1.1).

Proof: Take $k = 1$ in Theorem 3.1.

A special case of Corollary 5.1 was obtained by Nakao [11, p. 62]. In his work $N = 1$, and the damping term $Q(t, x, u, v)$ was assumed to have the separated form $h(t)\rho(x, v)$. Furthermore, he requires (in our notation) the following much stronger versions of (H1) and (AS),

$$0 \le f(x,u)\, u \le \text{Const.}\, |u|^{1+r/2}, \qquad h(t) \ge \text{Const.} = h_0 > 0,^1$$

$$C_1 |v|^m \le \rho(x,v)\, v \le C_2 |v|^m, \qquad m \in [2, 1 + r/2].$$

Under these assumptions we can take $p = 1 + r/2$ in (H1) and $q = m$, $\sigma(t) = C_1 h_0$, $\delta(t) = C_2 h(t)$ and $\omega(s) = \min\{s^{m-1},\ 1\}$ in (AS), in which case (5.1) reduces to

$$\liminf_{t\to\infty} \frac{1}{t^m} \int_0^t h(s) ds < \infty,$$

which is exactly condition (1.3) in [11, p. 57].

[1] Nakao [11, p. 62] actually writes only the weaker condition $0 \le f(x,u)\, u \le \text{Const.}\, (1 + |u|^{1+r/2})$, but this seems inaccurate. First, it is too weak in itself to obtain his earlier hypothesis (A_1). Second, he later adds continuity hypotheses which would imply (at the least) that

$$0 \le f(x,u)\, u \le \text{Const.}\, |u|\, (1 + |u|^{r/2}),$$

while finally in the earlier paper [10], which he refers to in the same context, it is explicitly required that $|f(x,u)| \le \text{Const.}\, |u|^{r/2}$, which is equivalent to the condition given here.

COROLLARY 5.2. *Assume that*

$$\liminf_{t\to\infty} \frac{1}{\log^m t}\int_1^t (\delta+\sigma^{1-m})\frac{ds}{s^m} < \infty. \tag{5.2}$$

Then (3.3) *holds for any strong solution* u *of* (1.1).

Proof: Take $k(t)=\min\{1,\ 1/t\}$ in Theorem 3.1.

It is possible to state a series of further corollaries, corresponding to the functions

$$k(t)=\min\{1,\ 1/t\log t\},\qquad k(t)=\min\{1,\ 1/t\log t\cdot\log|\log t|\},$$

and so forth, but we shall not pursue this here.

COROLLARY 5.3. *Suppose that*

$$(\delta+\sigma^{1-m})^{1/(1-m)}\in CBV(J),\qquad m>1, \tag{5.3}$$

with

$$\int_0^\infty (\delta+\sigma^{1-m})^{1/(1-m)}dt=\infty. \tag{5.4}$$

Then (3.3) *holds for any strong solution* u *of* (1.1).

Proof: Take $k=(\delta+\sigma^{1-m})^{1/(1-m)}$ in Theorem 3.1. Thus (3.1) is satisfied in view of (5.4), while (3.2) follows because

$$\lim_{t\to\infty}\left(\int_0^t k(s)ds\right)^{1-m}=0.$$

Remark. The choice $k=(\delta+\sigma^{1-m})^{1/(1-m)}$ can also be used in Theorem 4.1, provided that $k\in AC(J)$ and (4.2), (5.4) hold. In general this does not lead to a conclusion of particular interest, except when $\delta(t)\equiv\delta_0>0$, or $\sigma(t)\equiv\sigma_0>0$. For example, in the latter case, condition (4.2) becomes

$$|\delta'|\le \text{Const.}\,(1+\delta)^{1+\lambda/(m-1)}\qquad \text{a.e. in } J,$$

and (5.4) reduces to

$$\int_0^\infty \delta^{1/(1-m)}dt=\infty.$$

Corollary 5.4. *Suppose that $m = 2$ and either*

$$\liminf_{t\to\infty} \frac{1}{t^2}\int_0^t (\delta + 1/\sigma)ds < \infty,$$

or

$$1/(\delta + 1/\sigma) \in CBV(J), \qquad \int_0^\infty \frac{dt}{\delta + 1/\sigma} = \infty.$$

Then (3.3) *holds for any strong solution u of* (1.1).

The linear case

Consider the problem

$$\begin{cases} u_{tt} - \Delta u + a(t)t^\alpha u_t + V(x)u = 0 & \text{in } J \times \Omega, \\ u(t,x) = 0 & \text{on } J \times \partial\Omega, \end{cases} \tag{5.5}$$

where $J = [1,\infty)$, Ω is a bounded open subset of $\mathbb{R}^n$, $N = 1$, V is continuous in $\overline{\Omega}$, and $\alpha \in \mathbb{R}$. Assume moreover that there is a number $C > 0$ such that

$$1/C \le a(t) \le C \qquad \text{in } J. \tag{5.6}$$

Here

$$Q(t,x,u,v) = a(t)t^\alpha v, \qquad F(x,u) = \tfrac{1}{2}\, V(x)u^2,$$

so that (H1) is satisfied, and (AS) holds with

$$\delta(t) = 1/\sigma(t) = C\, t^{|\alpha|}, \qquad m = 2, \qquad \omega(s) = s.$$

Consider first the case $|\alpha| \le 1$. Then

$$1/(\delta + 1/\sigma) = t^{-|\alpha|}/2C \notin L^1(J).$$

Hence by Corollary 5.4 every strong solution u of (5.5) has the property that

$$\lim_{t\to\infty} Eu(t) = 0.$$

When $|\alpha| > 1$, Corollary 5.4 does not apply. In fact, if $\alpha < -1$ there exist oscillating solutions of (5.5) which *do not* approach zero as $t \to \infty$. To show this, consider solutions of (5.5) having the separated form

$$u(t,x) = w(t)\varphi(x), \tag{5.7}$$

where $\varphi = \varphi_k$ is an eigenfunction of $-\Delta + V(x)$ in Ω, with Dirichlet boundary conditions, having the corresponding eigenvalue $\mu = \mu_k > 0$. An easy calculation shows that w is a solution of the ordinary differential equation

$$w'' + a(t)t^\alpha w' + \mu\, w = 0, \qquad t \in J. \tag{5.8}$$

This equation satisfies hypotheses (5.1)–(5.4) of Theorem 5.1′ in [14], with $H(u,p) = \frac{1}{2}\, p^2$, and $\hat{\delta}(t) = 2a(t)t^\alpha$. Clearly $\hat{\delta} \in L^1(J)$ by (5.6). Therefore the only solution of (5.8) which approaches zero at ∞ is $w \equiv 0$.

This being shown, it is easy to argue that all non trivial solutions of (5.8) are oscillatory, with amplitude approaching a non–zero limit as $t \to \infty$. It may be noted that the behavior of solutions of (5.8) when $\alpha < -1$ is then essentially the same as for the *wave equation* itself in a bounded domain with zero boundary data.

When $\alpha > 1$, solutions of (5.5) again do not in general approach zero as $t \to \infty$, though their behavior is quite different from the case $\alpha < -1$. We say that a function

$$\psi = \psi(x) \in Y \equiv \operatorname{span}\{\varphi_k\}_{k=1}^\infty$$

is *attainable* if there exists a solution $u \in K$ of (5.5) such that

$$\lim_{t\to\infty} \|u(t) - \psi\|_{L^2} = 0. \tag{5.9}$$

Theorem 5.4. *Every function $\psi \in Y$ is attainable for problem* (5.5). *In turn, the set of attainable functions is dense in L^2.*

Proof: We first show that every eigenfunction φ_k of $-\Delta + V(x)$ in Ω, with eigenvalue $\mu_k > 0$, is attainable. For this purpose consider the function

$$u_k(t,x) = w_k(t)\varphi_k(x),$$

which satisfies (5.5) if and only if w_k is a solution of (5.8) with $\mu = \mu_k$. By [13, Theorem 4.4], since

$$1/a(t)t^\alpha \in L^1(J),$$

by (5.6), it follows that the set of attainable limits at ∞ of solutions of (5.8) is dense in $\mathbb{R}$. (Note that all solutions of (5.8) can be continued over all of J, because of linearity.) On the other hand, again since (5.8) is linear, the set of attainable limits for (5.8) must in fact be all of $\mathbb{R}$. Hence for an appropriate solution $w_k \not\equiv 0$ we get

$$\lim_{t\to\infty} \|u_k(t,\cdot) - \varphi_k\|_{L^2} = 0.$$

Finally, again from the linearity of (5.5), we obtain (5.9) for every $\psi \in Y$.

It is interesting to note that when $\alpha > 1$ the specific function

$$u(t,x) = \left(1 + \frac{\mu}{(\alpha-1)t^{\alpha-1}}\right)\varphi(x)$$

solves (5.5) corresponding to the choice

$$a(t) = a_\mu(t) = 1 + \frac{\mu}{(\alpha-1)t^{\alpha-1}} + \frac{\alpha}{t^{\alpha+1}};$$

note that (5.6) is then satisfied with $C = (\alpha^2 - 1 + \mu)/(\alpha - 1) > 1$. Here it is obvious that

$$\lim_{t\to\infty} \|u(t,\cdot) - \varphi\|_{L^2} = 0.$$

We can also treat the problem

$$(5.10) \qquad \begin{cases} u_{tt} - \Delta u + a(t)t^\alpha (\log t)^\beta u_t + V(x)u = 0 & \text{in } J \times \Omega, \\ u(t,x) = 0 & \text{on } J \times \partial\Omega. \end{cases}$$

The results are essentially the same, namely,

(*a*) If $|\alpha| < 1$; or if $\alpha = 1$, $\beta \le 1$; or if $\alpha = -1$, $\beta \ge -1$: then $Eu(t) \to 0$ as $t \to \infty$ for every strong solution u of (5.10).

(*b*) If $\alpha = -1$, $\beta < -1$; or if $\alpha < -1$: then there exist oscillating solutions of (5.10) which do not approach zero as $t \to \infty$.

(*c*) If $\alpha = 1$, $\beta > 1$; or if $\alpha > 1$: then every function $\psi \in Y$ is attainable at ∞, and in turn the set of attainable functions is dense in L^2.

6. Other operators

As noted in the introduction, the Laplace operator $-\Delta$ in (1.1) can be replaced by the polyharmonic operator $(-\Delta)^L$, $L \ge 1$, by the degenerate Laplace operator $-\Delta_s$, or, more generally, by

$$-\text{div}\, A(x, Du) = -\sum_{i=1}^n D_i A_{ij}(x, Du), \qquad j = 1, \dots, N,$$

where $u = (u_1, \dots, u_N)$ and $A \in C^1(\Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}^{n\times N})$. We treat each of these cases briefly.

1. The polyharmonic operator

Consider the problem

$$
(6.1)\qquad \begin{cases} u_{tt} + (-\Delta)^L u + Q(t,x,u,u_t) + f(x,u) = 0 & \text{in } J \times \Omega, \\ u(t,x) = 0,\ Du(t,x) = 0,\ \dots,\ D^{2(L-1)}u(t,x) = 0 & \text{on } J \times \partial\Omega. \end{cases}
$$

Define

$$
X = \left[W_0^{L,2}(\Omega)\right]^N, \qquad K' = C(J \to X) \cap C^1(J \to L^2),
$$

$$
\mathcal{D}_L \varphi = \begin{cases} D\Delta^j \varphi & \text{if } L = 2j+1 \\ \Delta^j \varphi & \text{if } L = 2j \end{cases},
$$

$$
K = \{\phi \in K' \ :\ \mathcal{F}\phi \text{ is well–defined and locally bounded on } J\},
$$

where $\mathcal{F}\phi$ is as in Section 2, and the total energy of ϕ is given by

$$
E\phi = E\phi(t) = \tfrac{1}{2}\|\phi'\|^2 + \tfrac{1}{2}\|\mathcal{D}_L\phi\|^2 + \mathcal{F}\phi.
$$

By a *strong solution* of (6.1) we mean a function $u \in K$ satisfying the following two conditions:

(A) *Distribution Identity*

$$
\langle u', \phi\rangle\Big]_0^t = \int_0^t \{\langle u', \phi'\rangle - \langle \mathcal{D}_L u, \mathcal{D}_L \phi\rangle - \langle Q(s,\cdot,u,u'), \phi\rangle - \langle f(\cdot,u), \phi\rangle\} ds
$$

for all $t \in J$ and $\phi \in K$.

(B) *Conservation Law*

(i) $$\langle Q(t,\cdot,u,u'), u'\rangle \in L^1_{\text{loc}}(J),$$

(ii) $$Eu\Big]_0^t = -\int_0^t \langle Q(s,\cdot,u,u'), u'\rangle ds \quad \text{for all } t \in J.$$

This definition is meaningful under hypotheses (H1)–(H3), provided r now denotes the Sobolev number for the space $\left[W_0^{L,2}(\Omega)\right]^N$, that is,

$$
(6.2)\qquad r = \frac{2n}{n-2L}, \qquad n > 2L
$$

($2 < r < \infty$ if $n = 1, \dots, 2L$).

Theorems 3.1 and 4.1 carry over equally, again with the only change that r is given by (6.2).

It almost goes without saying that (H1), (H2), and (AS) can be replaced by (H1)$'$, (H2)$'$, and (AS)$'$ when $n = 1, \dots, 2L-1$.

2. The degenerate Laplace operator

Consider the problem

$$\begin{cases} u_{tt} - \operatorname{div}(|Du|^{s-2}Du) + Q(t,x,u,u_t) + f(x,u) = 0 & \text{in } J \times \Omega, \\ u(t,x) = 0 \quad \text{on } J \times \partial\Omega, \end{cases} \tag{6.3}$$

where $s > 1$. Define $X = \left[W_0^{1,s}(\Omega)\right]^N$, $K' = C(J \to X) \cap C^1(J \to L^2)$, and

$$K = \{\phi \in K' \;:\; \mathcal{F}\phi \text{ is well–defined and locally bounded on } J\}.$$

The total energy is now given by

$$E\phi = E\phi(t) = \frac{1}{2}\|\phi'\|^2 + \frac{1}{s}\|D\phi\|^s_{L^s} + \mathcal{F}\phi.$$

The distribution identity then becomes

$$\langle u', \phi\rangle\Big]_0^t = \int_0^t \{\langle u', \phi'\rangle - \langle |Du|^{s-2}Du, D\phi\rangle - \langle Q(\tau,\cdot,u,u'),\phi\rangle - \langle f(\cdot,u),\phi\rangle\}d\tau$$

for all $t \in J$ and $\phi \in K$. The corresponding conditions (A) and (B) are then meaningful under assumptions (H1)–(H3), again provided that r is the Sobolev number for the space $\left[W_0^{1,s}(\Omega)\right]^N$, namely,

$$r = \frac{ns}{n-s}, \qquad n > s \tag{6.4}$$

($2 < r < \infty$ if $n = 1, \dots, s$). Theorems 3.1 and 4.1 carry over equally, *mutatis mutandis.*

3. The operator $\operatorname{div} A(x, Du)$

In order to formulate a conservation law for this case, it is necessary to assume that $A = A(x,w)$ is derivable from a potential, namely that

$$A(x,w) = \frac{\partial G}{\partial w}(x,w), \tag{6.5}$$

where $G \in C^2(\Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}_0^+)$. The total energy then has the form

$$Eu = Eu(t) = \tfrac{1}{2}\|u'\|^2 + \|G(x,Du)\|_{L^1} + \mathcal{F}u.$$

Furthermore, for an adequate definition of solution of the related Dirichlet problem, the potential $G(x,\cdot)$ must be coercive, in the sense that

$$|w|^s \le \text{Const.}\,(1+G(x,w)), \qquad s \ge 1, \tag{6.6}$$

for all $(x,w)\in\Omega\times\mathbb{R}^{n\times N}$. We can then take $X=\big[W_0^{1,s}(\Omega)\big]^N$ and K in the standard way. Finally, in order to carry out the proofs for asymptotic stability, in particular the preliminary result given in Lemma 3.4, we must assume that

$$0 \le G(x,w) \le \text{Const.}\,\big(A(x,w),w\big) \qquad \text{for all } (x,w)\in\Omega\times\mathbb{R}^{n\times N}. \tag{6.7}$$

With the assumptions (6.5)–(6.7) in hand, the previous arguments can be carried over to the case when the Laplace operator is replaced by $\operatorname{div} A(x,Du)$. We leave the details to the reader.

7. Non–autonomous higher order damping

Consider the problem

$$\begin{cases} u_{tt}-\Delta(u+\rho(t)u_t)+Q(t,x,u,u_t)+f(x,u)=0 & \text{in } J\times\Omega,\\ u(t,x)=0 \qquad \text{on } J\times\partial\Omega, \end{cases} \tag{7.1}$$

where $\rho\in L^1_{\rm loc}(J\to\mathbb{R}^+_0)$. Here we take

$$X=\big[W_0^{1,2}(\Omega)\big]^N, \qquad K'=C^1(J\to X),$$

and K in the usual way. Moreover, again,

$$Eu=Eu(t)=\tfrac12\|u'\|^2+\tfrac12\|Du\|^2+\mathcal{F}u.$$

By a *strong solution* of (7.1) we mean a function $u\in K$ satisfying:

(A) *Distribution Identity*

$$\langle u',\phi\rangle\big]_0^t=\int_0^t\{\langle u',\phi'\rangle-\langle Du,D\phi\rangle-\rho(s)\langle Du',D\phi\rangle-\langle Q(s,\cdot,u,u'),\phi\rangle-\langle f(\cdot,u),\phi\rangle\}ds$$

for all $t\in J$ and $\phi\in K$.

(B) *Conservation Law*

(i) $$\langle Q(t,\cdot,u,u'),u'\rangle\in L^1_{\rm loc}(J),$$

(ii) $$Eu\big]_0^t=-\int_0^t\{\langle Q(s,\cdot,u,u'),u'\rangle+\rho\|Du'\|^2\}ds \quad \text{for all } t\in J.$$

It is easy to see that this definition is meaningful when hypotheses (H1)–(H3) hold. Turning to the asymptotic stability of solutions of (7.1), we must amplify the discussion already given in Section 3 to take into account the term $\rho(t)\langle Du',D\phi\rangle$ in (A).

Theorem 7.1. *Let* (H1), (H3) *and* (AS) *hold. Suppose there exists* $k \in CBV(J \to \mathbb{R}_0^+)$ *such that*

$$k \notin L^1(J), \tag{7.2}$$

$$\liminf_{t\to\infty} \frac{\left(\int_0^t \rho\, k^2 ds\right)^{1/2} + \left(\int_0^t (\delta + \sigma^{1-m}) k^m ds\right)^{1/m}}{\int_0^t k ds} < \infty. \tag{7.3}$$

Then along any strong solution u *of* (7.1) *there holds*

$$\lim_{t\to\infty} Eu(t) = 0. \tag{7.4}$$

Before proving this result we require two further lemmas.

Lemma 7.1. *In addition to the conclusions* (3.4)–(3.6) *of Lemma 3.1, we also have*

$$\rho(t)\|Du'(t)\|^2 \in L^1(J). \tag{7.5}$$

The proof is the same as that of Lemma 3.1, with (3.6) and (7.5) deduced in exactly the same way as (3.6) before.

Lemma 7.2. *For all* $t \geq T \geq 0$ *we have*

$$\int_T^t k\rho\langle Du, Du'\rangle ds \leq \varepsilon(T) \left(\int_T^t \rho k^2 ds\right)^{1/2}, \tag{7.6}$$

where $\varepsilon(T) \to 0$ *as* $T \to \infty$.

Proof: By Schwarz' inequality and (3.5)

$$\langle Du, Du'\rangle \leq \text{Const.}\, \|Du'\|.$$

Hence, by integration from T to t, and another use of Schwarz' inequality,

$$\int_T^t k\rho\langle Du, Du'\rangle ds \leq \text{Const.} \left(\int_T^t \rho k^2 ds\right)^{1/2} \left(\int_T^t \rho\|Du'\|^2 ds\right)^{1/2}$$
$$\leq \varepsilon(T) \left(\int_T^t \rho k^2 ds\right)^{1/2},$$

where $\varepsilon(T) \to 0$ as $T \to \infty$ by (7.5).

Proof of Theorem 7.1: As in the proof of Theorem 3.1, we first treat the simpler case in which k is also of class $C^1(J)$. Suppose for contradiction that $\ell > 0$ in (3.12).

Consider the Lyapunov function

$$V(t) = \langle u', \phi\rangle, \qquad \phi = k(t)u \in K.$$

Hence by the distribution identity (A) above, for any $t \geq T \geq 0$, we have

$$V(s)\big]_T^t = \int_T^t \{k'\langle u, u'\rangle + 2k\|u'\|^2 - k(\|u'\|^2 + \|Du\|^2) - k\rho\langle Du, Du'\rangle - k\langle Q(s,\cdot,u,u'),u\rangle - k\langle f(\cdot,u),u\rangle\}ds.$$

Applying Lemmas 3.2–3.4, which continue to hold in the present case as one easily sees, together with Lemmas 7.1 and 7.2, we now obtain

$$\begin{aligned} V(s)\big]_T^t \leq \text{Const.} \int_T^\infty |k'|ds + 2\vartheta \int_T^t k ds + 2\varepsilon(T)C(\vartheta)\left(\int_0^t \sigma^{1-m}k^m ds\right)^{1/m} \\ - \alpha \int_T^t k ds + \varepsilon(T)\left\{1 + \left(\int_0^t \rho k^2 ds\right)^{1/2} + \left(\int_0^t \delta k^m ds\right)^{1/m}\right\}. \end{aligned} \tag{7.7}$$

The rest of the proof is the same as for Theorem 3.1, except of course (7.3) is used instead of (3.2).

We also have a related result of independent interest, valid under a much weaker form of (AS), that is,

$$|Q(t,x,u,v)| \leq \delta(t)\,[|v|^{m-1} + |v|^{q-1}], \tag{AS1}$$

where m, q, and δ are as in (AS).

Theorem 7.2. *Let* (H1)*,* (H3) *and* (AS1) *hold. Suppose there exists* $k \in CBV(J \to \mathbb{R}_0^+)$ *such that*

$$k \notin L^1(J), \qquad k \leq \text{Const.}\,\rho \quad \textit{in } J, \tag{7.8}$$

$$\liminf_{t\to\infty} \frac{\left(\int_0^t \rho\, k^2 ds\right)^{1/2} + \left(\int_0^t \delta\, k^m ds\right)^{1/m}}{\int_0^t k ds} < \infty. \tag{7.9}$$

Then (7.4) *holds.*

To begin with, observe that Lemmas 7.1, 7.2 and 3.2, 3.4 continue to hold when (AS) is replaced by (AS1). Lemma 3.3 no longer is valid, but in its place we use the following

LEMMA 7.3. *Let* $(7.8)_2$ *be satisfied. Then, for all* $t \geq T \geq 0$,

$$\int_T^t k \,\|u'\|^2 ds \leq \varepsilon(T),$$

where $\varepsilon(T) \to 0$ *as* $T \to \infty$.

Proof: By Poincaré's inequality and the fact that $u' \in C(J \to X)$, we have

$$\|u'\| \leq \text{Const.}\, \|Du'\|.$$

Consequently, for all $t \geq T \geq 0$,

$$\int_T^t k \,\|u'\|^2 ds \leq \text{Const.} \int_T^t \rho \,\|Du'\|^2 ds$$

by $(7.8)_2$. The assertion now follows from (7.5).

Proof of Theorem 7.2: We argue as in the proof of Theorem 7.1, until (7.7). By Lemmas 3.2, 3.4, and 7.1–7.3, we obtain, instead of (7.7),

$$V(s)\big]_T^t \leq \text{Const.} \int_T^\infty |k'| ds - \alpha \int_T^t k ds + \varepsilon(T) \left\{ 3 + \left(\int_0^t \rho k^2 ds \right)^{1/2} + \left(\int_0^t \delta k^m ds \right)^{1/m} \right\}.$$

The proof is now completed as before.

Remark. In case $n = 1$ we can allow slightly weaker hypotheses, of the type indicated in Sections 2 and 3. We leave the details to the reader.

Theorems 7.1 and 7.2 have a number of corollaries corresponding to those of Section 5.

COROLLARY 7.1. *Suppose*

$$\liminf_{t\to\infty} \left\{ \left(\frac{1}{t^2} \int_0^t \rho ds \right)^{1/2} + \left(\frac{1}{t^m} \int_0^t (\delta + \sigma^{1-m}) ds \right)^{1/m} \right\} < \infty. \tag{7.10}$$

Then (7.4) *holds.*

If $\rho(t) \geq \rho_0 > 0$ *in* J, *then the term* σ^{1-m} *can be dropped in* (7.10).

Proof: Take $k = 1$ in Theorems 7.1 and 7.2.

COROLLARY 7.2. *Suppose*

$$\liminf_{t\to\infty}\left\{\left(\frac{1}{\log^2 t}\int_1^t \rho\frac{ds}{s^2}\right)^{1/2}+\left(\frac{1}{\log^m t}\int_1^t(\delta+\sigma^{1-m})\frac{ds}{s^m}\right)^{1/m}\right\}<\infty. \tag{7.11}$$

Then (7.4) *holds.*

If $\rho(t)\ge\rho_0/t$ *in* J, *then the term* σ^{1-m} *can be dropped in* (7.11).

Proof: Take $k(t)=\min\{1,\ 1/t\}$ in Theorems 7.1 and 7.2.

A result corresponding to Corollary 5.3 can also be given. We leave the statement to the reader.

The condition that k be of bounded variation in Theorems 7.1 and 7.2 can be replaced by a differential relation corresponding to the results in Section 4. Exactly as there, we get the following analogue of Theorem 4.1.

THEOREM 7.3. *Let the assumptions of Theorem 7.1 hold, except that k is absolutely continuous rather than of bounded variation. Suppose also that the function ω in* (AS) *satisfies* (4.1), *and that k obeys* (4.2), (4.3). *Then* (7.4) *is valid.*

THEOREM 7.4. *Let the assumptions of Theorem 7.1 hold, except that k is absolutely continuous rather than of bounded variation. If also*

$$|k'|\le \text{Const.}\sqrt{\rho\, k}\qquad a.e.\ \ in\ J, \tag{7.12}$$

then (7.4) *holds.*

Proof: Again it is only necessary to estimate the term $k'\langle u,u'\rangle$ in the identity (3.18). We have

$$|k'\langle u,u'\rangle|\le \text{Const.}\sqrt{\rho\, k}\,\|u\|\cdot\|u'\|\le \text{Const.}\sqrt{\rho\, k}\,\|u\|\cdot\|Du'\|$$

by the Poincaré–Sobolev inequality. Then by (3.5) and Hőlder's inequality

$$\int_T^t|k'\langle u,u'\rangle|ds\le \text{Const.}\left(\int_T^t kds\right)^{1/2}\left(\int_T^t\rho\,\|Du'\|^2ds\right)^{1/2}\le\varepsilon(T)\left\{1+\int_T^t kds\right\}.$$

The remainder of the proof is the same as before.

THEOREM 7.5. *Let the assumptions of Theorem 7.2 hold, except that k is absolutely continuous rather than of bounded variation. If* (7.12) *is satisfied, then* (7.4) *holds.*

The proof is the same as that of Theorem 7.2, except that the term $k'\langle u,u'\rangle$ is estimated as in the proof of Theorem 7.4.

The linear case

Consider the problem

$$
(7.13)\qquad \begin{cases} u_{tt} - \Delta(u + b(t)t^{\beta}u_t) + a(t)t^{\alpha}u_t + V(x)u = 0 & \text{in } J \times \Omega, \\ u(t,x) = 0 \qquad \text{on } J \times \partial\Omega, \end{cases}
$$

where $N = 1$, $J = [1, \infty)$, Ω is a bounded open subset of $\mathbb{R}^n$, V is a continuous function on $\overline{\Omega}$, and α, $\beta \in \mathbb{R}$. We assume that

$$
1/C \le a(t) \le C, \qquad 1/D \le b(t) \le D \quad \text{in } J,
$$

where C, $D > 0$ are constants. Clearly (H1) is satisfied, and (AS) holds with $m = 2$, $\omega(s) \equiv s$, $\rho(t) = b(t)t^{\beta}$, and appropriate δ and σ

Then strong solutions of (7.13) *are asymptotically stable, that is,* (7.4) *holds, whenever either*

$$
\text{(i)} \quad \alpha \le 1 \textit{ and } |\beta| \le 1, \qquad \textit{or} \qquad \text{(ii)} \quad \beta \le 1 \textit{ and } |\alpha| \le 1.
$$

To see this, in case (i) use the second part of Corollary 7.2 with $m = 2$, $\delta(t) = C\,t$ and $(1/D\,t) \le \rho(t) \le D\,t$. Similarly, in case (ii) we use the first part of Corollary 7.1 with $m = 2$, $\delta(t) = C\,t$, $\sigma(t) = 1/C\,t$, and $\rho(t) \le D\,t$.

Note in fact that for the result (i) the condition $a(t) \ge 1/C$ is not needed – merely that $a(t) \ge 0$, while for (ii) the condition $b(t) \ge 1/D$ similarly is not necessary.

Exactly as in the case of problem (5.5), when $\alpha < -1$ and $\beta < -1$, there exist oscillating solutions of (7.13) which do not approach zero as $t \to \infty$. Indeed, it is enough to consider solutions having the separated form (5.7) and to observe that the function w then is a solution of the linear ordinary differential equation

$$
w'' + [a(t)t^{\alpha} + \mu\, b(t)t^{\beta}]w' + \mu\, w = 0.
$$

Similarly, when either $\alpha > 1$ or $\beta > 1$, the conclusion of Theorem 5.4 remains valid for (7.13), with essentially the same proof.

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