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LOCAL ASYMPTOTIC STABILITY FOR DISSIPATIVE WAVE SYSTEMS

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Abstract. We study the question of asymptotic stability, as time tends to infinity, of solutions of dissipative wave systems, governed by time–dependent nonlinear damping forces and by stongly nonlinear potential energies. This problem had been considered earlier for potential energies which arise from restoring forces, whereas here we allow as well for the effect of amplifying forces. Global asymptotic stability can then no longer be expected, and should be replaced by local stability.

The conclusions are related to and supplement earlier work of Payne and Sattinger [7], who treated the nondissipative case, and of Hale [1], who showed the existence of connected global attractors.

§1. Introduction

We study the problem of asymptotic stability, as time tends to infinity, of solutions of dissipative wave systems, governed by time dependent nonlinear damping forces and subject to the action of strongly nonlinear potential energies. In earlier papers [2, 4–9] the question of global asymptotic stability was considered when the potential energy arises from a restoring force. Here both restoring and amplifying effects are allowed, so that global asymptotic stability is generally no longer to be expected, and an expanded treatment is required.

More precisely we consider vectorial solutions $u = u(t, x)$, $u : I \times \Omega \to \mathbb{R}^N$, with $N \geq 1$, of the problem

$$(1.1)\qquad \begin{cases} u_{tt} - \Delta u + Q(t, x, u_t) + f(x, u) = 0, & (t, x) \in I \times \Omega, \\ u(t, x) = 0, & (t, x) \in I \times \partial\Omega, \\ u(0, \cdot) \in H^1_0, & u_t(0, \cdot) \in L^2, \end{cases}$$

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where $I=[0,\infty)$ and Ω is a bounded open subset of $\mathbb{R}^n$, $n\ge 1$,

$$H_0^1=\left[H_0^1(\Omega)\right]^N,\qquad L^\rho=\left[L^\rho(\Omega)\right]^N,\quad \rho\ge 1,$$

and we assume

$$Q\in C(I\times\Omega\times\mathbb{R}^N\to\mathbb{R}^N),\qquad f\in C(\Omega\times\mathbb{R}^N\to\mathbb{R}^N).$$

The function Q represents a *nonlinear damping*, so that

$$\big(Q(t,x,v),v\big)\ge 0\quad\text{for all arguments } t,x,v, \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$. Concerning the forcing term f, we suppose there is a real valued potential F such that

$$f(x,u)=\frac{\partial F}{\partial u}(x,u),\qquad F(x,0)\equiv 0; \tag{1.3}$$

of course $(1.3)_1$ is automatic when $N=1$.

Solutions of (1.1) are required to be in the set

$$K=C(I\to H_0^1)\cap C^1(I\to L^2).$$

a precise definition of solution, in an appropriate distribution sense, will be postponed to Section 2.

It is convenient, however, to anticipate here the structural assumptions on Q and f which we require for our main results. For simplicity in stating the hypotheses, we first consider dimensions $n\ge 2$, and refer to Section 2 for the special case $n=1$. Let r denote the Sobolev exponent for the space H_0^1, namely

$$r=\frac{2n}{n-2}\qquad\text{when } n\ge 3,$$

and r any real number satisfying $r>2$ when $n=2$.

(A1) There are functions $\sigma=\sigma(t)$, $\omega=\omega(\tau)$ such that

$$\big(Q(t,x,v),v\big)\ge\sigma(t)\omega(|v|)\quad\text{for all arguments } t,x,v, \tag{1.4}$$

where $\omega\in C(\mathbb{R}_0^+\to\mathbb{R}_0^+)$ is increasing, $\omega(0)=0$ and $\omega(\tau)=\tau^2$ for $\tau\ge 1$, while $\sigma\ge 0$ and $1/\sigma\in L_{\rm loc}^{\nu-1}(I)$ for some exponent $\nu>1$.

Moreover, there are exponents m, q satisfying

$$2\le m<q\le r$$

such that for all arguments t, x, v,

$$|Q(t,x,v)| \le d_1(t,x)^{1/m}\big(Q(t,x,v),v\big)^{1/m'} + d_2(t,x)^{1/q}\big(Q(t,x,v),v\big)^{1/q'}, \tag{1.5}$$

where m' and q' are the Hőlder conjugates of m and q and

$$\delta_1(t) = \|d_1(t,\cdot)\|_{r/(r-m)}, \qquad \delta_2(t) = \begin{cases} \|d_2(t,\cdot)\|_{r/(r-q)}, & \text{if } q < r, \\ \|d_2(t,\cdot)\|_\infty, & \text{if } q = r. \end{cases}$$

When $N = 1$, or more generally if the function Q is *tame*, i.e., there is a constant $\gamma \ge 1$ such that

$$|Q(t,x,v)| \cdot |v| \le \gamma\,\big(Q(t,x,v),v\big) \qquad \text{for all arguments } t,x,v,$$

then (1.5) can be written equivalently in the form

$$|Q(t,x,v)| \le \text{ Const.}\,\{d_1(t,x)|v|^{m-1} + d_2(t,x)|v|^{q-1}\},$$

see Remark 1 in Section 5. This last condition shows more clearly the rôle of d_1 and d_2 in giving an upper bound for $|Q(t,x,v)|$.

(A2) There is a constant $C > 0$ and an exponent p, $1 < p \le r$, such that

$$|f(x,u)| \le C\big(1 + |u|^{p-1}\big) \qquad \text{in } \Omega \times \mathbb{R}^N.$$

(A3) There is an auxiliary function $k \in C^1(I \to \mathbb{R}_0^+)$, $k \not\equiv 0$, such that

$$\lim_{t\to\infty} \frac{\int_0^t |k'(s)|ds}{\int_0^t k(s)ds} = 0, \tag{1.6}$$

$$\liminf_{t\to\infty} \frac{\Big(\int_0^t \delta_1 k^m ds\Big)^{1/m} + \Big(\int_0^t \delta_2 k^q ds\Big)^{1/q} + \Big(\int_0^t \sigma^{1-\nu} k^\nu ds\Big)^{1/\nu}}{\int_0^t k ds} < \infty. \tag{1.7}$$

Condition (A3) has the purpose of preventing the non–autonomous terms δ_1, δ_2 in the damping estimates from being too large, and the term σ from being too small, since otherwise one cannot in general obtain asymptotic stability for the system (1.1). Some cases of allowable functions σ, δ_1, δ_2 are given in Section 5.

Note. The condition $k \in C^1(I \to \mathbb{R}_0^+)$ in (A3) can be weakened to $k \in AC(I \to \mathbb{R}_0^+)$, that is $k \in W^{1,1}_{\text{loc}}(I \to \mathbb{R}_0^+)$, as follows from an easy approximation argument (see also the Appendix of [10]).

Let μ_0 be the first eigenvalue of $-\Delta$ in Ω, with zero Dirichlet boundary conditions, and put

$$\|\cdot\|_\rho = \|\cdot\|_{L^\rho}, \quad \rho > 1; \qquad \|\cdot\| = \|\cdot\|_2.$$

Then under the structural conditions (A1)–(A3) we have

Theorem 1. *Suppose there is a number* $\mu < \mu_0$ *such that*

$$\big(f(x,u),u\big) \geq -\mu|u|^2 \qquad in\ \Omega \times \mathbb{R}^N. \tag{1.8}$$

Then for every strong solution $u \in K$ *of* (1.1)*, in the sense defined in Section* 2*, we have*

$$\lim_{t\to\infty} \big\{\|u_t(t,\cdot)\| + \|Du(t,\cdot)\|\big\} = 0. \tag{1.9}$$

That is, the rest state $(0,0)$ *is a globally attracting set in* $H^1_0 \times L^2$ *for* (1.1).

The proof will be given in Section 3. Theorem 1 is related to a well–known stability result of Hale, contained in Theorem 4.8.11 of [1]. Problem (1.1) was treated there in the scalar case $N = 1$ and for dimensions $n \geq 3$ (though explicitly described only for $n = 3$). Moreover, the damping term Q was required to be *autonomous*, that is $Q = Q(v)$, as well as the force f independent of x, or $f = f(u)$. In addition, it was assumed that

$$Q \in C^1(\mathbb{R} \to \mathbb{R}), \qquad Q(0) = 0, \qquad 0 < \alpha \leq Q'(v) \leq \beta \tag{1.10}$$

for all $v \in \mathbb{R}$, while

$$f \in C^1(\mathbb{R} \to \mathbb{R}), \qquad f(0) = 0, \qquad |f'(u)| \leq C\big(1 + |u|^{p-2}\big) \tag{1.11}$$

for every $u \in \mathbb{R}$, with $p < 1 + r/2$. Under the further main restriction

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} \geq 0 \tag{1.12}$$

the existence of a *connected global attractor* $A \subset H^1_0 \times L^2$ was obtained for (1.1).

To compare this result with Theorem 1, we first note that (A1) holds for (1.10) with $\sigma(t) \equiv \alpha$, $\omega(\tau) = \tau^2$, $\nu = 2$, $d_1(t,x) \equiv \beta^2/\alpha$ and $d_2(t,x) \equiv 0$; also assumption (A2) is obviously satisfied since $p < 1 + r/2 < r$. Finally (A3) is automatic with $k(t) \equiv 1$, since d_1, d_2, and σ are constants. Thus, with the exception of (1.8), the conditions in [1] are special cases of those here.

On the other hand, neither (1.8) nor (1.12) implies the other, since (1.12) is a stronger condition for $|u|$ large, while (1.8) is stronger otherwise.

Our second theorem is related to Hale's result in a different way, with (1.8) required *only for sufficiently small* u.

Theorem 2. *Suppose*

$$\liminf_{u\to 0} \frac{\big(f(x,u),u\big)}{|u|^2} \geq -\overline{\mu}, \tag{1.13}$$

where $\overline{\mu} < \mu_0$. *If* u *is a strong solution of* (1.1) *with sufficiently small initial data* $\|Du(0,\cdot)\|$, $\|u_t(0,\cdot)\|$, *then* (1.9) *continues to hold.*

In particular the rest state $(0,0)$ *is a locally attracting set in* $H_0^1 \times L^2$ *for the system* (1.1).

When (1.1) is non–dissipative ($Q \equiv 0$), a related conclusion was obtained by Payne and Sattinger [7, pages 294–295], though the fact that Q vanishes in their work precludes strict asymptotic stability.

Finally, it can be expected that Hale's theorem holds even when (1.12) is replaced by

$$(1.12)' \qquad \liminf_{|u|\to\infty} \frac{f(u)}{u} \geq -\overline{\mu},$$

with $\overline{\mu} < \mu_0$ as above. In this case conditions $(1.12)'$ and (1.13) exhibit a striking duality, and the corresponding conclusions an interesting dichotomy.

Theorem 2 will be proved in Section 4. Some examples and applications of our results are given in Section 5.

§2. Definition of solution

Consider the *total energy* $E\phi$ of a vector field $\phi \in K$, that is

$$(2.1) \qquad E\phi = E\phi(t) = \tfrac{1}{2}\|\phi_t(t,\cdot)\|^2 + \tfrac{1}{2}\|D\phi(t,\cdot)\|^2 + \mathcal{F}\phi(t),$$

where $\mathcal{F}\phi$ is the *potential energy of the field*, namely

$$\mathcal{F}\phi(t) = \int_\Omega F(x,\phi(t,x))dx.$$

It is easy to see that $\mathcal{F}\phi$ is well–defined in I. Indeed, by (A2),

$$(2.2) \qquad |F(x,u)| = \left| \int_0^1 \big(f(x,su),u\big)ds \right| \leq C\left(|u| + \frac{1}{p}|u|^p\right), \qquad 2 < p \leq r,$$

in $\Omega \times \mathbb{R}^N$. Hence, by Sobolev's embedding theorem, we have $F(\cdot,\phi(t,\cdot)) \in L^1(\Omega)$ for all $\phi \in K$ and $t \in I$, and in particular

$$(2.3) \qquad |\mathcal{F}\phi(t)| \leq \text{ Const.}\big(\|D\phi(t,\cdot)\| + \|D\phi(t,\cdot)\|^p\big), \quad t \in I.$$

It is convenient to introduce the elementary bracket pairing $\langle\cdot,\cdot\rangle$ in $\Omega\subset\mathbb{R}^n$

$$\langle\varphi,\psi\rangle=\langle\varphi(t,\cdot),\psi(t,\cdot)\rangle=\int_\Omega\big(\varphi(t,x),\psi(t,x)\big)dx,$$

where $\varphi,\,\psi:\,I\times\Omega\to\mathbb{R}^N$, this being a well–defined real function of time for all $t\in I$ such that $(\varphi,\psi)\in L^1$.

For every $\phi\in K$, we put

$$\mathcal{D}\phi(t)=\int_\Omega\big(Q(t,x,\phi_t(t,x)),\phi_t(t,x)\big)dx=\langle Q(t,\cdot,\phi_t),\phi_t\rangle,$$

this being a well–defined non–negative function on I, with possibly infinite values. In fact, since $\big(Q(t,x,v),v\big)\geq 0$ by (1.2), and since Q is continuous in its variables, the integrand is, for each fixed t, a non–negative measurable function of x, as required.

Definition. We say that $u\in K$ is a *strong solution of the system* (1.1) if

(a) $\mathcal{D}u\in L^1_{\rm loc}(I)$;

(b) $Eu(t)+\int_0^t\mathcal{D}u(s)ds$ is a non–increasing function in I;

(c) $$\langle u_t,\phi\rangle\big]_0^t=\int_0^t\{\langle u_t,\phi_t\rangle-\langle Du,D\phi\rangle-\langle Q(s,\cdot,u_t),\phi\rangle-\langle f(\cdot,u),\phi\rangle\}ds$$

for all $t\in I$ and $\phi\in K$.

The function $\mathcal{D}u$ represents the rate of dissipation of energy of any solution u of (1.1), so that (a) expresses the natural requirement that the total dissipation over any finite time interval should be finite. Condition (b) is a generalization of the usual conservation law for the system (1.1), for which the function in (b) is *constant*, i.e. the initial value of the energy. We emphasize that energy conservation, or more generally some appropriately formulated but weaker version of this law, such as condition (b), is an essential attribute of any reasonable solution. In fact, standard existence theorems (see for example [1], [2], [6], as well as the recent work [12] in which the case $N>1$ is considered) all yield solutions satisfying both strict conservation of energy as well as condition (c). At the same time, since in that work stronger conditions than (1.4), (1.5) were necessary for the proofs, it seems important to allow in (b) as great a weakening of energy conservation as possible while at the same time still retaining its main attributes. See also the comments in [8, Section 2 and Remark 4 in Section 4]. The distribution identity (c) is of course the natural setting for solutions of (1.1), in that classical solutions of this problem need not generally exist.

It is important to note that (a) and (b) imply that *Eu is non–increasing in I*, since $\int_0^t\mathcal{D}u(s)ds$ is non–decreasing in I because $\mathcal{D}u$ is non-negative.

From the definition of K it is clear that the identity (c) is meaningful provided that

$$\langle f(\cdot, u(t,\cdot)), \phi(t,\cdot)\rangle,\ \langle Q(t,\cdot,u_t(t,\cdot)), \phi(t,\cdot)\rangle \in L^1_{\mathrm{loc}}(I). \tag{2.4}$$

By (A2) and Hőlder's inequality we immediately get

$$|\langle f(\cdot, u(t,\cdot)), \phi(t,\cdot)\rangle| \le C\big(\|\phi(t,\cdot)\|_1 + \|u(t,\cdot)\|_p^{p-1} \cdot \|\phi(t,\cdot)\|_p\big).$$

On the other hand, by Sobolev's embedding theorem (since Ω is bounded)

$$u,\ \phi \in C(I \to L^\rho), \qquad 1 \le \rho \le r,$$

so that $\langle f(\cdot,u), \phi(t,\cdot)\rangle \in L^\infty_{\mathrm{loc}}(I)$ and $(2.4)_1$ holds.

To obtain $(2.4)_2$ we observe by (1.5) that, for $q < r$,

$$\begin{aligned}\|Q(t,\cdot,u_t(t,\cdot))\|_{r'} &\le \|d_1(t,\cdot)^{1/m}\big(Q(t,\cdot,u_t),u_t\big)^{1/m'}\|_{r'} + \|d_2(t,\cdot)^{1/q}\big(Q(t,\cdot,u_t),u_t\big)^{1/q'}\|_{r'} \\ &\le \|d_1(t,\cdot)\|^{1/m}_{r/(r-m)}\|\big(Q(t,\cdot,u_t),u_t\big)\|_1^{1/m'} \\ &\qquad + \|d_2(t,\cdot)\|^{1/q}_{r/(r-q)}\|\big(Q(t,\cdot,u_t),u_t\big)\|_1^{1/q'} \\ &= \delta_1(t)^{1/m}\mathcal{D}u(t)^{1/m'} + \delta_2(t)^{1/q}\mathcal{D}u(t)^{1/q'}\end{aligned}$$

by Hőlder's inequality and the definition of $\mathcal{D}u$. When $q = r$ we get in essentially the same way

$$\|Q(t,\cdot,u_t(t,\cdot))\|_{r'} \le \delta_1(t)^{1/m}\mathcal{D}u(t)^{1/m'} + \delta_2(t)^{1/r}\mathcal{D}u(t)^{1/r'}.$$

In turn, again by Hőlder's inequality, for each $t \in I$ and for $2 \le m < q \le r$

$$\begin{aligned}\int_0^t |\langle Q(s,\cdot,u_t),\phi\rangle| ds &\le \int_0^t \|Q(s,\cdot,u_t(s,\cdot))\|_{r'} \cdot \|\phi(s,\cdot)\|_r ds \\ &\le \max_{[0,t]} \|\phi(s,\cdot)\|_r \left[\left(\int_0^t \delta_1(s)ds\right)^{1/m}\left(\int_0^t \mathcal{D}u(s)ds\right)^{1/m'}\right. \\ &\qquad \left. + \left(\int_0^t \delta_2(s)ds\right)^{1/q}\left(\int_0^t \mathcal{D}u(s)ds\right)^{1/q'}\right] \le C(t)\end{aligned} \tag{2.5}$$

since $\delta_1,\ \delta_2 \in L^1_{\mathrm{loc}}(I)$ by (A1) and $\mathcal{D}u \in L^1_{\mathrm{loc}}(I)$ by (a). Consequently $(2.4)_2$ is proved, and condition (c) is meaningful.

To give a particular example of a function Q satisfying (A1), one may take

$$Q(t,x,v) = d_1(t,x)|v|^{m-2}v + d_2(t,x)|v|^{q-2}v, \qquad d_1,\ d_2 \in C(I \times \Omega \to \mathbb{R}^+_0),$$

with m, q, d_1 and d_2 as in (A1). Obviously

$$\big(Q(t,x,v),v\big) = d_1(t,x)|v|^m + d_2(t,x)|v|^q,$$

so that

$$\begin{aligned}|Q(t,x,v)| &\le d_1(t,x)|v|^{m-1} + d_2(t,x)|v|^{q-1}\\ &= d_1(t,x)^{1/m}\big(d_1(t,x)^{1/m'}|v|^{m-1}\big) + d_2(t,x)^{1/q}\big(d_2(t,x)^{1/q'}|v|^{q-1}\big)\\ &\le d_1(t,x)^{1/m}\big(Q(t,x,v),v\big)^{1/m'} + d_2(t,x)^{1/q}\big(Q(t,x,v),v\big)^{1/q'},\end{aligned}$$

and (1.5) holds. Similarly (1.4) is valid with $\omega(\tau)=\tau^q$ for $\tau\in[0,1]$, $\omega(\tau)=\tau^2$ for $\tau\ge 1$, and

$$\sigma(t) = \inf_{\Omega}\{d_1(t,x)+d_2(t,x)\}.$$

Naturally one now requires $1/\sigma \in L^{\nu-1}_{\mathrm{loc}}(I)$. In particular the zero set of σ can be at most of measure zero.

We conclude this first part with the following easy fact:

Let $k\in C^1(I)$ be a non–negative, non–trivial function satisfying (1.6)*. Then $k\notin L^1(I)$.*

Indeed, suppose the contrary. Then (1.6) obviously gives $\int_0^\infty |k'(s)|ds = 0$, and so $k'=0$ in I. Hence $k\equiv$Const., and in turn $k\equiv 0$ since $k\in L^1(I)$ by the assumption of contradiction. But k is non–trivial by hypothesis and this completes the proof.

The case $n=1$.

Here the hypothesis (A2) can be weakened to the form

(A2)′ $$|f(x,u)|\le g(u), \qquad g\in C(\mathbb{R}^N \to \mathbb{R}^+_0),$$

for all $(x,u)\in\Omega\times\mathbb{R}^N$.

For all u, $\phi\in K$ and $t\in I$ then $\mathcal{F}\phi$ is well–defined and locally bounded on I for all $\phi\in K$, while also

$$|\langle f(\cdot,u(t,\cdot)),\phi(t,\cdot)\rangle| \le c_1\|\phi(t,\cdot)\|_{L^\infty},$$

where

$$c_1 = c_1(t) = |\Omega| \sup_{w\in B(t)} g(w),$$

and $B(t)=\{w\in\mathbb{R}^N \ : \ |w|\le \|u(t,\cdot)\|_{L^\infty}\}$.

Indeed, because $n=1$ and $\phi(t,\cdot)\in H^1_0$ for all $t\in I$, we get

$$\|\phi(t,\cdot)\|_{L^\infty} \le \sqrt{|\Omega|/2}\ \|D\phi(t,\cdot)\| \in C(I),$$

with a similar estimate for $\|u(t,\cdot)\|_{L^\infty}$. Thus, $|\langle f(\cdot,u(t,\cdot)),\phi(t,\cdot)\rangle| \in L^1_{\rm loc}(I)$ since c_1 is locally bounded on I.

Remark. Condition (A2)$'$ is automatic if $f \in C(\overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N)$, or if f does not depend on x.

For (A1) we now define $\delta_1(t) = \|d_1(t,\cdot)\|_1$ and $\delta_2(t) = \|d_2(t,\cdot)\|_1$, but otherwise no changes are necessary.

§3. Proof of Theorem 1

We apply the principal ideas of [8], with various modifications due to the fact that the forcing term f is no longer assumed to be of restoring type; that is we now allow $\big(f(x,u),u\big)$ to take negative values. In addition, condition (A1) is stated more generally than the corresponding hypotheses of Section 3 of [8], which requires further modifications in the discussion.

The proof is given in a series of lemmas. We assume throughout that the conditions (A1)–(A2) are satisfied, and that $u \in K$ is a given strong solution of (1.1) in I, in the sense of Section 2.

Lemma 3.1. *Let* (1.8) *hold. Then*

$$Eu(t) \geq \tfrac{1}{2}\|u_t(t,\cdot)\|^2 + \tfrac{1}{2}\left(1-\frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 \quad \text{in } I. \tag{3.1}$$

Moreover,

$$\|u_t(t,\cdot)\|,\ \|Du(t,\cdot)\|,\ \|u(t,\cdot)\|_r \in L^\infty(I), \tag{3.2}$$

and

$$\mathcal{D}u \in L^1(I). \tag{3.3}$$

Proof. By (1.8) we have $F(x,u) \geq -\frac{1}{2}\mu|u|^2$, so that $\mathcal{F}u(t) \geq -\frac{1}{2}\mu\|u(t,\cdot)\|^2$. Hence (3.1) follows from (1.6) and the Poincaré inequality. Conditions (3.2) are then immediate from the fact that Eu is non–increasing in I, so $Eu(t) \leq Eu(0)$.

Finally, (3.3) is a consequence of (b); indeed $Eu \geq 0$ in I by (3.1), and $\mathcal{D}u \geq 0$ in I, giving

$$0 \leq \int_I \mathcal{D}u(s)ds \leq Eu(0).$$

Lemma 3.2. *Let $k \in L^\infty_{\mathrm{loc}}(I)$ and $k \geq 0$. Then, for any $T \in I$ and $t \geq T$, we have*

$$\int_T^t k\,|\langle Q(s,\cdot,u_t),u\rangle|ds \leq \varepsilon(T)\left[\left(\int_0^t \delta_1 k^m ds\right)^{1/m} + \left(\int_0^t \delta_2 k^q ds\right)^{1/q}\right], \tag{3.4}$$

where $\varepsilon(T) \to 0$ as $T \to \infty$.

Proof. Exactly as in (2.5)

$$\int_T^t k|\langle Q(s,\cdot,u_t),u\rangle|ds \leq \max_{[T,t]} \|u(s,\cdot)\|_r \left[\left(\int_T^t \delta_1 k^m ds\right)^{1/m}\left(\int_T^t \mathcal{D}u(s)ds\right)^{1/m'} + \left(\int_T^t \delta_2 k^q ds\right)^{1/q}\left(\int_T^t \mathcal{D}u(s)ds\right)^{1/q'}\right].$$

From $(3.2)_3$ there is a number $B_0 > 0$ such that $\|u(s,\cdot)\|_r \leq B_0$ for all $s \in I$. Thus (3.4) holds with

$$\varepsilon(T) = B_0\left[\left(\int_T^\infty \mathcal{D}u(s)ds\right)^{1/m'} + \left(\int_T^\infty \mathcal{D}u(s)ds\right)^{1/q'}\right],$$

and $\varepsilon(T) \to 0$ as $T \to \infty$ by (3.3).

Lemma 3.3. *Suppose $k \in L^\infty_{\mathrm{loc}}(I)$ and $k \geq 0$. Let ϑ be a given positive constant. Then there exists $C(\vartheta) > 0$ such that for all $t \geq T \geq 0$*

$$\int_T^t k\|u_t(s,\cdot)\|^2 ds \leq \vartheta \int_T^t k ds + \varepsilon(T)C(\vartheta)\left(\int_T^t \sigma^{1-\nu}k^\nu ds\right)^{1/\nu}, \tag{3.5}$$

where $\varepsilon(T) \to 0$ as $T \to \infty$, and $\nu > 1$ is the exponent in (A1).

Proof. The proof is essentially the same as for Lemma 3.3 in [8]. For the sake of completeness we reproduce the details.

For fixed $s \in I$, define

$$\Omega_1 = \Omega_1(s) = \{x \in \Omega \ : \ |u_t(s,x)| \leq \sqrt{\vartheta = /|\Omega|}\},$$
$$\Omega_2 = \Omega_2(s) = \{x \in \Omega \ : \ |u_t(s,x)| > \sqrt{\vartheta/|\Omega|}\}.$$

Clearly

$$\int_{\Omega_1} |u_t(s,x)|^2 dx \leq \vartheta. \tag{3.6}$$

For $x \in \Omega_2$ we write

$$\int_{\Omega_2} |u_t(s,x)|^2 dx = \int_{\Omega_2} \frac{|u_t(s,x)|^2}{\omega(|u_t(s,x)|)} \cdot \omega(|u_t(s,x)|) dx, \tag{3.7}$$

where ω is the function appearing in (A1). Now

$$\sup_{\tau \geq \sqrt{\vartheta/|\Omega|}} \frac{\tau^2}{\omega(\tau)} = \max_{\sqrt{\vartheta/|\Omega|} \leq \tau \leq 1} \frac{\tau^2}{\omega(\tau)} = \Lambda(\vartheta), \tag{3.8}$$

since ω is continuous and positive for $\tau > 0$, and $\omega(\tau) == \tau^2$ for $\tau \geq 1$. From (3.7), (3.8) and (1.4) we obtain

$$\int_{\Omega_2} |u_t(s,x)|^2 dx \leq \frac{\Lambda(\vartheta)}{\sigma(s)} \int_\Omega \big(Q(s,x,u_t(s,x)), u_t(s,x)\big) dx = \frac{\Lambda(\vartheta)}{\sigma(s)} \mathcal{D}u(s).$$

In turn, since $\|u_t(s,\cdot)\| \leq$Const. in I by (3.2), we have

$$\begin{aligned} k \int_{\Omega_2} |u_t(s,x)|^2 dx &\leq \text{Const.}\, k \left(\int_{\Omega_2} |u_t(s,x)|^2 dx \right)^{1/\nu'} \\ &\leq \text{Const.}\, [\Lambda(\vartheta)]^{1/\nu'} (\sigma^{1-\nu} k^\nu)^{1/\nu} \mathcal{D}u(s)^{1/\nu'}, \end{aligned}$$

where $\nu > 1$ is the exponent in (A1). Thus (3.6) yields

$$k\|u_t(s,\cdot)\|^2 \leq \vartheta k + \text{Const.}\, [\Lambda(\vartheta)]^{1/\nu'} (\sigma^{1-\nu} k^\nu)^{1/\nu} \mathcal{D}u(s)^{1/\nu'}.$$

The required result now follows by integration from T to t and another use of Hőlder's inequality; in particular, we can take $C(\vartheta) = [\Lambda(\vartheta)]^{1/\nu'}$, and

$$\varepsilon(T) = \text{Const.} \left(\int_T^\infty \mathcal{D}u(s) ds \right)^{1/\nu'} \to 0 \qquad \text{as } T \to \infty$$

by (3.3).

Since Eu is non–increasing and non–negative, there exists $\ell \geq 0$ such that

$$\lim_{t\to\infty} Eu(t) = \ell. \tag{3.9}$$

Lemma 3.4. *Suppose $\ell > 0$ in (3.9). Then there exists $\alpha = \alpha(\ell) > 0$ such that*

$$\|u_t(t,\cdot)\|^2 + \|Du(t,\cdot)\|^2 + \langle f(\cdot,u), u(t,\cdot)\rangle \geq \alpha \qquad \text{on } I. \tag{3.10}$$

Proof. The proof is similar to that of Lemma 3.4 of [8]. Divide I into the two sets

$$I_1 = \{t \in I \ : \ \mathcal{F}u(t) \leq \ell/2\}, \qquad I_2 = I \setminus I_1 = \{t \in J \ : \ \mathcal{F}u(t) > \ell/2\}.$$

For $t \in I_1$

$$\|u_t(t,\cdot)\|^2 + \|Du(t,\cdot)\|^2 \geq 2[\ell - \mathcal{F}u(t)] \geq \ell.$$

Consequently by (1.8)

$$\begin{aligned} \|u_t(t,\cdot)\|^2 + \|Du(t,\cdot)\|^2 + \langle f(\cdot,u), u(t,\cdot)\rangle &\geq \|u_t(t,\cdot)\|^2 + \|Du(t,\cdot)\|^2 - \mu\|u(t,\cdot)\|^2 \\ &\geq \|u_t(t,\cdot)\|^2 + \left(1 - \frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 \\ &\geq \left(1 - \frac{\mu}{\mu_0}\right)\ell. \end{aligned} \tag{3.11}$$

On the other hand, by (2.2) we see that in I_2

$$\tfrac{1}{2}\ell < \mathcal{F}u(t) \leq C(\|u(t,\cdot)\|_1 + \tfrac{1}{p}\|u(t,\cdot)\|_p^p).$$

Since $H_0^1 \subset L^\rho$ for $1 \leq \rho \leq r$, one has

$$\|u(t,\cdot)\|_\rho \leq B_\rho\,\|Du(t,\cdot)\|, \tag{3.12}$$

where B_ρ is the Sobolev constant for the embedding, depending on n, ρ, $|\Omega|$. Consequently,

$$\begin{aligned} \tfrac{1}{2}\ell < \mathcal{F}u(t) &\leq C_1(\|Du(t,\cdot)\| + \|Du(t,\cdot)\|^p) \\ &\leq 2C_1 \begin{cases} \|D(u(t,\cdot)\| & \text{if } \|Du(t,\cdot)\| \leq 1 \\ \|D(u(t,\cdot)\|^p & \text{if } \|Du(t,\cdot)\| > 1 \end{cases} \end{aligned}$$

for an appropriate constant C_1, depending on C, B_1, B_p and p. Hence

$$\|Du(t,\cdot)\| \geq \min\left\{\frac{\ell}{4C_1},\ \left(\frac{\ell}{4C_1}\right)^{1/p}\right\} = C_2(\ell). \tag{3.13}$$

Therefore, using (3.11), we find

$$\begin{aligned} \|u_t(t,\cdot)\|^2 + \|Du(t,\cdot)\|^2 + \langle f(\cdot,u), u(t,\cdot)\rangle &\geq \left(1 - \frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 \\ &\geq \left(1 - \frac{\mu}{\mu_0}\right)C_2^2(\ell). \end{aligned}$$

Thus property (3.10) holds in I with

$$\alpha = \alpha(\ell) = \left(1 - \frac{\mu}{\mu_0}\right) \min\{\ell,\ C_2^2(\ell)\}.$$

Proof of Theorem 1. Suppose for contradiction that $\ell > 0$ in (3.9). Define a second Lyapunov function by

$$V(t) = k(t)\langle u, u_t\rangle = \langle u_t, \phi\rangle, \qquad \phi = k(t)u.$$

Since $k \in C^1(I)$ and $u \in K$, we have $\phi_t = (k\,u)_t = k'u + ku_t$, so that obviously $\phi \in K$. It now follows from the distribution identity (c) in Section 2 that for any $t \geq T \geq 0$

$$V(s)\big]_T^t = \int_T^t \{k'\langle u, u_t\rangle + 2k\|u_t\|^2 - k(\|u_t\|^2 + \|Du\|^2 + \langle f(\cdot, u), u\rangle) - k\langle Q(s, \cdot, u_t), u\rangle\} ds.$$

We now estimate the right hand side of this identity.

First

$$|\langle u(t,\cdot), u_t(t,\cdot)\rangle| \leq \|u(t,\cdot)\| \cdot \|u_t(t,\cdot)\| \leq L \qquad \text{in } I \tag{3.14}$$

by (3.2). Applying Lemmas 3.2–3.4 to the remaining terms, we then obtain

$$V(s)\big]_T^t \leq L\int_T^t |k'|ds + 2\vartheta \int_T^t kds + 2\varepsilon(T)C(\vartheta)\left(\int_0^t \sigma^{1-\nu}k^\nu ds\right)^{1/\nu} - \alpha\int_T^t kds + \varepsilon(T)\left\{\left(\int_0^t \delta_1 k^m ds\right)^{1/m} + \left(\int_0^t \delta_2 k^q ds\right)^{1/q}\right\}.$$

Let $A(t)$ denote the numerator function in (1.7). Then, by taking $\vartheta = \vartheta(\ell) = \alpha/4$, we get

$$V(s)\big]_T^t \leq L\int_T^t |k'|ds - \frac{\alpha}{2}\int_T^t kds + \varepsilon(T)[2\,C(\vartheta) + 1]A(t). \tag{3.15}$$

By (1.7) there is a sequence $t_i \nearrow \infty$ and a number $M > 0$ such that

$$A(t_i) \leq M\int_0^{t_i} kds \qquad \text{for all } i. \tag{3.16}$$

We now take T so large that

$$\varepsilon(T)[2\,C(\vartheta) + 1]M \leq \alpha/4;$$

together with (3.15) and (3.16), this gives for all $t_i \geq T$

$$V(t_i) \leq V(T) + L\int_T^{t_i} |k'|ds + \frac{\alpha}{2}\int_0^T kds - \frac{\alpha}{4}\int_0^{t_i} kds. \tag{3.17}$$

Finally, by (3.14)

$$V(t_i) = k(t_i)\langle u(t_i,\cdot), u_t(t_i,\cdot)\rangle \geq -L\left\{k(0) + \int_0^{t_i} |k'(s)|ds\right\}.$$

Combining this with (3.17) gives

$$0 \leq \text{ Const. } + 2L\int_0^{t_i} |k'(s)|ds - \frac{\alpha}{4}\int_0^{t_i} k(s)ds \leq \text{ Const. } - \frac{\alpha}{8}\int_0^{t_i} k(s)ds,$$

provided that i is chosen even larger, if necessary. Letting $i \to \infty$ and using the fact that $k \notin L^1(I)$ by (1.6) – see the end of Section 2 – we obtain the required contradiction. Hence $\ell = 0$ in (3.9), completing the proof.

§4. Proof of Theorem 2

Again we start with a series of lemmas, assuming always that (A1) and (A2) are satisfied. Let $u \in K$ be a given strong solution of (1.1) in I. In the proof we can clearly assume without loss of generality that $p > 2$.

Lemma 4.1. *Let* (1.13) *hold. Then there exist constants* $c > 0$ *and* μ, *with* $\overline{\mu} < \mu < \mu_0$, *such that*

$$\big(f(x,u), u\big) \geq -\mu|u|^2 - c|u|^p \quad \text{in } \Omega \times \mathbb{R}^N, \tag{4.1}$$

$$\mathcal{F}u(t) \geq -\frac{\mu}{2}\|u(t,\cdot)\|^2 - \frac{c}{p}\|u(t,\cdot)\|_p^p \quad \text{in } I, \tag{4.2}$$

and

$$Eu(t) \geq \tfrac{1}{2}\|u_t(t,\cdot)\|^2 + \tfrac{1}{4}\left(1 - \frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 + a\|u(t,\cdot)\|_p^2 - \frac{c}{p}\|u(t,\cdot)\|_p^p, \tag{4.3}$$

in I, *where*

$$a = \frac{1}{4B_p^2}\left(1 - \frac{\mu}{\mu_0}\right)$$

and B_p is the Sobolev constant defined in (3.12).

Proof. The inequalities (4.1) and (4.2) are immediate consequences of (1.13) and (A2). By (2.1), (4.2) and Poincaré's inequality

$$Eu(t) \geq \tfrac{1}{2}\|u_t(t,\cdot)\|^2 + \tfrac{1}{2}\left(1 - \frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 - \frac{c}{p}\|u(t,\cdot)\|_p^p.$$

Hence, in turn, by the Sobolev embedding (3.12) we get (4.3).

Define

$$\Sigma = \{(\lambda, E) \in \mathbb{R}^2 \ : \ 0 \leq \lambda < \lambda_1, \ 0 \leq E < E_1\}, \tag{4.4}$$

where

$$\lambda_1 = \left(\frac{a}{c}\right)^{1/(p-2)}, \qquad E_1 = \left(2 - \frac{1}{p}\right) a\lambda_1^2,$$

and a, c are the numbers introduced in Lemma 4.1.

Lemma 4.2. *Let $\lambda(t) = \|u(t,\cdot)\|_p$. If $\big(\lambda(0), Eu(0)\big) \in \Sigma$, then*

$$\big(\lambda(t), Eu(t)\big) \in \Sigma \quad \textit{for all } t \in I. \tag{4.5}$$

Moreover

$$Eu(t) \geq \tfrac{1}{2}\|u_t(t,\cdot)\|^2 + \tfrac{1}{4}\left(1 - \frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 \quad \textit{in } I. \tag{4.6}$$

Proof. Assume that $\big(\lambda(0), Eu(0)\big) \in \Sigma$. Then, since Eu is non–increasing in I, we have $E_1 > Eu(0) \geq Eu(t)$. Hence by (4.3) and another use of the Sobolev embedding inequality, there results, for all $t \in I$,

$$2a\lambda(t)^2 - \frac{c}{p}\lambda(t)^p \leq Eu(t) < E_1. \tag{4.7}$$

Hence $\lambda(t) \neq \lambda_1$ for any $t \in I$, since

$$E_1 = 2a\lambda_1^2 - \frac{c}{p}\lambda_1^p.$$

Now, $\lambda(t) \in C(I)$ and $\lambda(0) < \lambda_1$, so that $\lambda(t) < \lambda_1$ for all $t \in I$.

To obtain (4.5) and (4.6) it remains to note (since $p > 2$) that

$$a\lambda^2 - \frac{c}{p}\lambda^p \geq 0 \qquad \text{whenever } 0 \leq \lambda < \lambda_1,$$

and to use (4.3) and (4.7).

Lemma 4.3. *If* $\big(\lambda(0), Eu(0)\big) \in \Sigma$*, then*

$$\|Du(t,\cdot)\|^2 + \langle f(\cdot, u(t,\cdot)), u(t,\cdot)\rangle \geq \tfrac{1}{2}\left(1 - \frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 \quad \text{in } I. \tag{4.8}$$

Proof. This is obtained almost exactly as (4.6) in Lemma 4.2. By (4.1), together with the Poincaré and Sobolev inequalities,

$$\|Du(t,\cdot)\|^2 + \langle f(\cdot, u(t,\cdot)), u(t,\cdot)\rangle \geq \tfrac{1}{2}\left(1 - \frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 + a\lambda(t)^2 - c\lambda(t)^p.$$

Consequently (4.8) holds since $0 \leq \lambda(t) < \lambda_1$ and $a\lambda^2 - c\lambda^p \geq 0$ if $0 \leq \lambda < \lambda_1$.

Remark. Lemmas 4.2 and 4.3 are easily visualized using the two–dimensional phase plane (λ, E) shown in Figure 1. In particular, by (4.7) any point $\big(\lambda(t), Eu(t)\big)$ on the trajectory of a solution $u \in K$ must lie above the curve

$$\Gamma: \quad E = 2a\lambda^2 - \frac{c}{p}\,\lambda^p.$$

In turn, if $\big(\lambda(t), Eu(t)\big) \in \Sigma$, this point must in fact be in the part Σ' of Σ which lies above Γ.

The region Σ' is shaded in Figure 1, with λ_1 defined by $a\lambda_1^2 - c\lambda_1^p = 0$. As we shall see, if $\big(\lambda(0), Eu(0)\big) \in \Sigma'$, then $\lim_{t\to\infty} Eu(t) = 0$.

Let $\Sigma_0 = \{(\lambda, E) \in \mathbb{R}^2 \ : \ 0 \leq \lambda < \lambda_0,\ 0 \leq E < E_0\}$, where λ_0 and E_0 are as shown in Figure 1. Then, reasoning as above, if $\big(\lambda(0), Eu(0)\big) \in \Sigma_0$, it follows that also $\big(\lambda(t), Eu(t)\big) \in \Sigma_0$. One might conjecture that, in this case, also $Eu(t)$ must approach 0 as t tends to infinity, but this *does not* appear to be the case.

A diagram similar to Figure 1 appears also in [7, page 274]. In the terminology of [7] the region Σ_0', that is the part of Σ_0 above the curve Γ, is a *potential well.*

Figure 1. The phase–plane (λ, E). The curves Γ and $E = a\lambda^2 - c\lambda^p$ are drawn to scale for the case $p = 4$.

Lemma 4.4. *If* $\big(\lambda(0), Eu(0)\big) \in \Sigma$*, then*

$$\|u_t(t,\cdot)\|,\ \|Du(t,\cdot)\|,\ \|u(t,\cdot)\|_r \in L^\infty(I), \tag{4.9}$$

and

$$\mathcal{D}u \in L^1(I). \tag{4.10}$$

Proof. The properties (4.9) are obvious from (4.6). Condition (4.10) then follows exactly as (3.3) in Lemma 3.1, since $Eu(t) \geq 0$ in I by (4.5).

The estimates (3.4) and (3.5) can be now obtained as before from (4.9), provided that $\big(\lambda(0), Eu(0)\big) \in \Sigma$. Moreover Eu is then non–negative and non–increasing in I, so that there exists $\ell \geq 0$ such that (3.9) holds.

Lemma 4.5. *Let* $\big(\lambda(0), Eu(0)\big) \in \Sigma$ *and suppose* $\ell > 0$ *in* (3.9)*. Then for some* $\alpha = \alpha(\ell)$ *the inequality* (3.10) *continues to hold.*

Proof. As before we divide I into the two sets

$$I_1 = \{t \in I\ :\ \mathcal{F}u(t) \leq \ell/2\}, \qquad I_2 = I \setminus I_1.$$

In I_1 we have

$$\|u_t(t,\cdot)\|^2 + \|Du(t,\cdot)\|^2 \geq \ell.$$

Then by (4.8)

$$\begin{aligned}\|u_t(t,\cdot)\|^2 + \|Du(t,\cdot)\|^2 + \langle f(\cdot,u), u(t,\cdot)\rangle &\geq \|u_t(t,\cdot)\|^2 + \tfrac{1}{2}\left(1 - \frac{\mu}{\mu_0}\right)\|Du(t,\cdot)\|^2 \\ &\geq \tfrac{1}{2}\left(1 - \frac{\mu}{\mu_0}\right)\ell.\end{aligned}$$

For $t \in I_2$ the required lower estimate is obtained exactly as in Lemma 3.4, using (4.8) and (3.13). Thus (3.10) is valid provided that

$$\alpha = \alpha(\ell) = \tfrac{1}{2}\left(1 - \frac{\mu}{\mu_0}\right)\min\{\ell,\ C_2^2(\ell)\}$$

(which is exactly one–half the previous value for $\alpha(\ell)$).

Proof of Theorem 2. Let $\big(\lambda(0), Eu(0)\big) \in \Sigma$. Using Lemmas 4.4–4.5 and the estimates (3.4) and (3.5), we derive as in the proof of Theorem 1 that

$$\lim_{t\to\infty} Eu(t) = 0. \tag{4.11}$$

In turn, (4.11) and (4.6) imply (1.9). It remains to be shown that if the data $\|u_t(0,\cdot)\|$ and $\|Du(0,\cdot)\|$ are sufficiently small, then $\big(\lambda(0), Eu(0)\big) \in \Sigma$. But $\lambda(0) < \lambda_1$ if $\|Du(0,\cdot)\|$ is suitably small, while equally from the definition of Eu and (2.3)

$$Eu(0) \le \tfrac{1}{2}\|u_t(0,\cdot)\|^2 + \tfrac{1}{2}\|Du(0,\cdot)\|^2 + \text{Const.}\big(\|Du(0,\cdot)\| + \|Du(0,\cdot)\|^p\big).$$

This implies $Eu(0) < E_1$ for suitably small data. Finally, since $0 \le \lambda(0) < \lambda_1$ it follows that $a\lambda(0)^2 - c\lambda(0)^p/p \ge 0$ and so $Eu(0) \ge 0$ by (4.3). This completes the proof.

§5. Examples and remarks

As a special case of our results, suppose that in (A1)

$$\delta_1(t) \le \text{Const.}\, t^{\beta_1}, \qquad \delta_2(t) \le \text{Const.}\, t^{\beta_2}, \qquad \sigma(t) \ge \text{Const.}\, t^{\gamma}$$

as $t \to \infty$. Take $k \equiv 1$ in (A3), in which case (1.6) is automatic and (1.7) reduces to the requirements

$$\beta_1 \le m-1, \qquad \beta_2 \le q-1, \qquad \gamma \ge -1.$$

The last condition follows independently of the choice of the exponent $\nu > 1$ in (A1).

Various further examples can easily be reformulated in the present setting, see e.g. [8, Section 5].

The principal ideas can also be extended to other hyperbolic systems, for example,

$$(|u_t|^{\ell-2}u_t)' - \operatorname{div}(|Du|^{s-2}Du) + Q(t,x,u_t) + f(x,u) = 0, \tag{5.1}$$

where $\ell > 1$, $s > 1$ and, instead of (1.5),

$$s \le m < q \le r_s, \qquad r_s = \frac{ns}{n-s} \quad \text{when } n > s.$$

Of course in (A2) the upper bound r for p should now be replaced by r_s.

In a recent paper [3] the question of blow–up for solutions of (1.1) was treated under the assumption (1.5) with $d_2 \equiv 0$. Condition (A2) was also required with $2 \leq m < p$ *and with*

$$2\,F(x,u) - \big(f(x,u),u\big) \geq c_1|u|^p \tag{5.2}$$

for some constant $c_1 > 0$.

A function f which simultaneously satisfies the conditions of the present paper and those of [3] is given by

$$f(u) = -\mu\, u - c|u|^{p-2}u, \tag{5.3}$$

where $\mu < \mu_0$, $c > 0$, and p is such that

$$2 \leq m < p \leq r.$$

Clearly (5.3) satisfies (5.2) with $c_1 = c(1-2/p) > 0$.

In order to illustrate the relations between the stability properties here and the blow–up results in [3], suppose $\delta_1(t) \leq$ Const. t^β, $\delta_2(t) \equiv 0$, $\sigma(t) \geq$ Const. t^γ as $t \to \infty$. Then local asymptotic stability holds for $\beta \leq m-1$ and $\gamma \geq -1$, as noted above. On the other hand, blow–up occurs when $-\infty < \gamma \leq \beta \leq m-1$ and $Eu(0) < 0$.

Note by (4.7) that if $Eu(0) < 0$ then necessarily $2a\lambda(0)^2 - c\lambda(0)^p/p < 0$, that is

$$\lambda(0) > (2ap/c)^{1/(p-2)} = \lambda_2,$$

see Figure 1. On the other hand, for local asymptotic stability it is enough to have $0 \leq \lambda(0) < \lambda_1 = (a/c)^{1/(p-2)}$ and $0 \leq Eu(0) < E_1$. Of course $\lambda_1 < \lambda_2$.

A related dichotomy between stability and blow–up was discussed by Payne and Sattinger [7] in somewhat similar terms (see pages 292–295). Of course, in their work $Q \equiv 0$, so that their results and ours cannot be compared in detail; we remark, however, that the variationally defined number d in [7] corresponds roughly to the number E_0 here.

Concluding remarks. 1. In the introduction we noted that when Q is tame, then (1.5) can be written in the equivalent form

$$|Q(t,x,v)| \leq \tilde{d}_1(t,x)|v|^{m-1} + \tilde{d}_2(t,x)|v|^{q-1}. \tag{5.4}$$

To show this, first suppose that (1.5) holds. Then by Young's inequality

$$\begin{aligned}\big(Q(t,x,v),v\big) &\leq |Q(t,x,v)|\cdot|v| \\ &\leq d_1(t,x)^{1/m}\big(Q(t,x,v),v\big)^{1/m'}|v| + d_2(t,x)^{1/q}\big(Q(t,x,v),v\big)^{1/q'}|v| \\ &\leq 4^{m/m'}d_1(t,x)|v|^m + \tfrac{1}{4}\big(Q(t,x,v),v\big) + 4^{q/q'}d_2(t,x)|v|^q + \tfrac{1}{4}\big(Q(t,x,v),v\big).\end{aligned}$$

Hence

$$\big(Q(t,x,v),v\big) \le 2^{2m-1}d_1(t,x)|v|^m + 2^{2q-1}d_2(t,x)|v|^q. \tag{5.5}$$

Now, using the tameness condition, we get (5.4) with $\tilde d_1 = 2^{2m-1}\gamma\, d_1$ and $\tilde d_2 = 2^{2q-1}\gamma\, d_2$.

On the other hand, if (5.4) holds then we fix $(t,x,v) \in I\times\Omega\times\mathbb{R}^N$ and distinguish two cases: $\tilde d_1(t,x)|v|^m \ge \tilde d_2(t,x)|v|^q$ and $\tilde d_1(t,x)|v|^m < \tilde d_2(t,x)|v|^q$. In the first case, by (5.4) and the tameness condition, we have

$$\begin{aligned}|Q(t,x,v)| &= |Q(t,x,v)|^{1/m}|Q(t,x,v)|^{1/m'} \le 2^{1/m}\tilde d_1(t,x)^{1/m}|v|^{(m-1)/m}|Q(t,x,v)|^{1/m'}\\ &\le \gamma^{1/m'}[2\tilde d_1(t,x)]^{1/m}\big(Q(t,x,v),v\big)^{1/m'}.\end{aligned}$$

Similarly, in the second case, we find

$$|Q(t,x,v)| \le \gamma^{1/q'}[2\tilde d_2(t,x)]^{1/q}\big(Q(t,x,v),v\big)^{1/q'}.$$

Combining the results in the two cases, we obtain (1.5), with $d_1 = 2\gamma^{m-1}\tilde d_1$ and $d_2 = 2\gamma^{q-1}\tilde d_2$, as required.

The relation (5.5) should be also emphasized as a further bound for $\big(Q(t,x,v),v\big)$ in terms of d_1 and d_2, which does not use the tameness condition.

2. Condition (b) in the definition of strong solution in Section 2 can be weakened to

$$Eu(t) + \int_0^t \mathcal{D}u(s)ds \le Eu(0) \qquad \text{in } I, \tag{b$'$}$$

provided the conclusion (1.9) is replaced by

$$\liminf_{t\to\infty}\big\{\|u_t(t,\cdot)\| + \|Du(t,\cdot)\|\big\} = 0. \tag{1.9$'$}$$

The condition (b)$'$ was suggested to us by Prof. Herbert Koch. It was also used as a principal assumption in [3] in place of (b).

3. Condition (A1) need not hold in the entire interval $I = [0,\infty)$, but in fact can be restricted to a measurable *control subset* $J \subset I$, see [10, Section 3]. Theorems 1 and 2 then need to be revised only in two places. First, in (A3) we assume that

$$k = 0 \quad \text{in } I\setminus J,$$

and second, in (1.7) the integrals in the numerator should be restricted to the set $[0,t]\cap J$. The proofs are almost exactly the same.

The importance of control subsets in studying asymptotic stability is illustrated in [11], in the context of ordinary differential systems.

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