Differential and Integral Equations

REMARKS ON LYAPUNOV STABILITY

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Abstract. We prove a number of asymptotic stability theorems of Lyapunov type for first order ordinary differential systems, extending and generalizing previous work on the subject.

1. Introduction. A recent monograph on Lyapunov theory [1] recalls the main trends in the basic theory, and presents a series of representative theorems concerning asymptotic stability (Theorems 1.1.2–1.1.5). Another pair of representative Lyapunov theorems appears in an earlier note of Yoshizawa [10]. Here we shall show that these results fall into two basic categories, in each of which one can state a single general conclusion containing the previous results as special cases.

The principal ingredients of Lyapunov's direct method consist first in the choice of an appropriate Lyapunov inequality corresponding to the differential system under consideration, and then in the application of the following

Lemma. *Let $\Phi : [T,\infty) \to \mathbb{R}$ be an absolutely continuous function which is bounded below and satisfies*

$$\Phi'(t) \leq -p(t) \qquad a.e.\ \text{in } I = [T,\infty),$$

where $p \in L^1_{\mathrm{loc}}(I)$. Then

$$\limsup_{t\to\infty} \int_T^t p(s)\,ds < \infty.$$

Our results are thus based in each case on the formulation of an abstract Lyapunov inequality in a suitable context. In Section 2 we

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consider a first category of results, corresponding, roughly speaking, to cases when the Lyapunov function is known to approach a limit as $t \to \infty$. In Section 3 we treat, conversely, cases where the Lyapunov function neither approaches a limit nor is bounded away from zero. Lastly, in Section 4 we consider the situation when the attracting set is larger than a single point.

Both [1] and [10] contain various examples and applications; thus we can concentrate here on the abstract theory. We note finally that the results quoted from [1] are based primarily on an earlier survey paper of Lakshmikantham [2].

2. First set of results. The concept of a *wedge* plays a fundamental rôle throughout, a wedge being a continuous non-decreasing function $W\colon[0,\infty) \to [0,\infty)$ with $W(0) = 0$ and $W(s) > 0$ for $s > 0$. All functions W are wedges in what follows, possibly changing from one case to the next without specific mention.

For simplicity in terminology, we shall say here that a *Lyapunov function* for a differential system

$$x' = f(t,x), \qquad f \in C(I \times \mathbb{R}^N; \mathbb{R}^N), \tag{2.1}$$

is a continuously differentiable function $V\colon I \times \mathbb{R}^N \to \mathbb{R}$, with the corresponding *definition*

$$V'_{(f)} \equiv V_t + V_x \cdot f\colon \quad I \times \mathbb{R}^N \to \mathbb{R}.$$

In particular, if $x = x(t)$ is a solution of (2.1), then

$$\frac{d}{dt} V(t, x(t)) \equiv V'_{(f)}(t, x(t)).$$

In the sequel V, $\hat{V}$ will always denote Lyapunov functions for the system (2.1), and B a correspondingly given open subset of $\mathbb{R}^N$ containing 0. The ancestor of the developments in this section is the following result (see [1, Theorem 1.1.2]).

Theorem A. *Suppose that V is bounded above and below by wedges, that is*

$$W_1(|x|) \le V(t,x) \le W_2(|x|), \tag{2.2}$$

and that

$$V'_{(f)}(t,x) \le -W(|x|) \tag{2.3}$$

for $(t,x) \in I\times B$*. Then every solution* $x = x(t)$ *of* (2.1) *which ultimately lies in* B *approaches zero as* $t \to \infty$*.*

The proof is immediate, once one observes that $V(t, x(t))$ tends to a non-negative finite limit ℓ as $t \to \infty$, for any solution $x = x(t)$ which ultimately lies in B. For if $\ell = 0$ then by $(2.2)_1$ we get $x(t) \to 0$ as $t \to \infty$. Otherwise, if $\ell > 0$, then by $(2.2)_2$ and (2.3) there follows

$$\frac{dV}{dt}(t, x(t)) \leq -c, \quad c = \text{const.} > 0.$$

Taking $p(t) \equiv c$ in the lemma now yields a contradiction.

If Ω is a positively invariant set of (2.1), and $\Omega \subset B$, then the conclusion of Theorem A can be written, alternately, that any solution of (2.1) which enters Ω must approach zero as $t \to \infty$. A corresponding extension of our later results is always possible, without further remark.

The ideas used in Theorem A apply also in the next result, essentially due to the present authors [8] and to Leoni [4]. In this result, and in what follows in the rest of the paper, the customary assumptions $V(t, 0) \equiv 0$ and $V(t, x) \geq 0$ are not required (see also [3, page 59, condition (i)]), but are replaced by the weaker condition (2.8) below.

It also almost goes without saying that the domain of $f(t, \cdot)$ in (2.1) could be smaller than $\mathbb{R}^N$ without significantly altering the conclusions.

Finally, we note that the continuity condition for f is used only to assure that $V(t, x(t))$ is continuously differentiable along solutions of (2.1). Thus if f is of *Carathéodory type*, and solutions of (2.1) are required only to be absolutely continuous functions satisfying (2.1) almost everywhere, then $V(t, x(t))$ is itself absolutely continuous on solutions since V is of class C^1, and the results of the paper remain valid with only minor technical changes.

Theorem 1. *Suppose that*

$$\hat{V}(t, x) \geq 0 \tag{2.4}$$

and

$$V'_{(f)}(t, x) \leq \varphi(t) - W(\hat{V})k(t) \tag{2.5}$$

for $(t, x) \in I \times B$*, where*

$$k \geq 0, \quad \varphi, k \in L^1_{\text{loc}}(I), \quad k \notin L^1(I). \tag{2.6}$$

Assume also that

$$\liminf_{t\to\infty} \int_T^t \varphi(s)\,ds \Big/ \int_T^t k(s)\,ds \le 0. \tag{2.7}$$

Let $x = x(t)$ *be a solution of* (2.1) *which ultimately lies in* B *and also satisfies*

$$\liminf_{t\to\infty} V(t, x(t)) > -\infty. \tag{2.8}$$

Then

$$\liminf_{t\to\infty} \hat{V}(t, x(t)) = 0.$$

It is important to note the relation between the (canonical) conclusion of Theorem A, namely

$$\lim_{t\to\infty} x(t) = 0,$$

and that of Theorem 1,

$$\liminf_{t\to\infty} \hat{V}(t, x(t)) = 0.$$

Indeed, if we add to the hypotheses of Theorem 1 that

$$\lim_{t\to\infty} \hat{V}(t, x(t)) \quad \text{exists,} \tag{2.8$'$}$$

then obviously the conclusion can be strengthened to

$$\lim_{t\to\infty} \hat{V}(t, x(t)) = 0.$$

Finally, if (2.4) is replaced by

$$\hat{V}(t, x(t)) \ge W_1(|x|) \tag{2.4$'$}$$

for some wedge W_1, then, even more, we get $\lim_{t\to\infty} x(t) = 0$.

Theorem A is an immediate corollary of the above remark in the special case

$$\varphi \equiv 0, \qquad k \equiv 1, \qquad \hat{V} \equiv V.$$

In particular, $(2.2)_2$ and (2.3) then imply (2.5), while $(2.8)'$ is a consequence of $(2.2)_1$ and (2.3), and (2.7) is trivially satisfied.

Proof of Theorem 1. We argue by contradiction, and assume the conclusion false. Then by (2.4) there exists $\ell > 0$ such that

$$\liminf_{t\to\infty} \hat{V}(t, x(t)) > \ell.$$

Then by (2.5)

$$\frac{dV}{dt}(t, x(t)) \le \varphi(t) - W(\ell)k(t) \tag{2.9}$$

for all t sufficiently large, say $t \ge T_1$. Moreover, by (2.8) the function $\Phi(t) \equiv V(t, x(t))$ is bounded below in I. Now set $p(t) = -\varphi(t) + W(\ell)k(t)$ for $t \in I$. Clearly $p \in L^1_{\text{loc}}(I)$; hence from (2.9) and our principal lemma we therefore get

$$\limsup_{t\to\infty} \int_{T_1}^{t} p(s)\, ds < \infty. \tag{2.10}$$

On the other hand, by (2.7), there is a sequence $t_i \nearrow \infty$ such that for all i

$$\int_T^{t_i} \varphi(s)\, ds \le \tfrac{1}{2} W(\ell) \int_T^{t_i} k(s)\, ds.$$

Therefore

$$\int_T^{t_i} p(s)\, ds \ge \tfrac{1}{2} W(\ell) \int_T^{t_i} k(s)\, ds,$$

which diverges to ∞ as $i \to \infty$ since $k \notin L^1(I)$. This contradicts (2.10) and completes the proof.

Theorem 1 can easily be rephrased to allow for the possibility that the function $\varphi = \varphi(t)$ depends on the particular solution $x = x(t)$.

There are, furthermore, has several fairly obvious corollaries of Theorem 1, which are of sufficient interest to state separately.

Corollary 1. *Suppose* (2.4)–(2.6) *are satisfied and that* $\varphi \in L^1(I)$. *Let* $x = x(t)$ *be a solution of* (2.1) *which ultimately lies in* B *and also satisfies* (2.8). *Then the conclusion of Theorem 1 continues to hold.*

This follows from the fact that (2.7) is a consequence of the conditions $\varphi \in L^1(I)$, $k \notin L^1(I)$.

The special case of Corollary 1 in which $\hat{V} = V$ is also of interest, since (2.8) is then a direct consequence of (2.4). Moreover, by (2.5) it is clear that $V(t, x(t))$ has a limit as $t \to \infty$; hence the conclusion can be strengthened to the form

$$\lim_{t\to\infty} V(t, x(t)) = 0.$$

Corollary 2. *Suppose that* (2.4) *is satisfied and that*

$$V'_{(f)}(t,x) \le \psi(t) + \varepsilon(t)\{\varphi(t) + k(t)\} - W(\hat{V})k(t)$$

for $(t,x) \in I \times B$*, where* φ *and* k *satisfy* (2.6),

$$\liminf_{t\to\infty} \int_T^t \varphi(s)\,ds \Big/ \int_T^t k(s)\,ds < \infty, \tag{2.11}$$

$\psi \in L^1(I)$*, and* ε *is a continuous function on* I *with* $\varepsilon(t) \to 0$ *as* $t \to \infty$.

Let $x = x(t)$ *be a solution of* (2.1) *which ultimately lies in* B *and also satisfies* (2.8)*. Then the conclusion of Theorem 1 holds.*

Proof. Define

$$\tilde{\varphi}(t) = \psi(t) + \varepsilon(t)\{\varphi(t) + k(t)\}, \quad t \in I.$$

Then $\tilde{\varphi} \in L^1_{\text{loc}}(I)$, while moreover for any $T_0 \ge T$ and $t > T_0$ we have

$$\int_{T_0}^t \tilde{\varphi}(s)\,ds \Big/ \int_{T_0}^t k(s)\,ds \le \int_I \psi(s)\,ds \Big/ \int_{T_0}^t k(s)\,ds + \tilde{\varepsilon}(T_0)\left\{\int_{T_0}^t \varphi(s)\,ds \Big/ \int_{T_0}^t k(s)\,ds + 1\right\},$$

where $\tilde{\varepsilon}(T_0) = \sup_{t \ge T_0} \varepsilon(t)$. Thus in turn, since $\tilde{\varphi}$, $k \in L^1_{\text{loc}}(I)$ and $k \notin L^1(I)$,

$$\liminf_{t\to\infty} \int_T^t \tilde{\varphi}(s)\,ds \Big/ \int_T^t k(s)\,ds = \liminf_{t\to\infty} \int_{T_0}^t \tilde{\varphi}(s)\,ds \Big/ \int_{T_0}^t k(s)\,ds \le \text{Const.}\,\tilde{\varepsilon}(T_0),$$

using (2.11) and the fact that $\psi \in L^1(I)$. Since $\tilde{\varepsilon}(T_0)$ can be made arbitrarily small by choosing T_0 sufficiently large, we infer that $\tilde{\varphi}$ satisfies (2.7). The corollary now follows at once from Theorem 1, with φ replaced by $\tilde{\varphi}$.

The next result contains Theorem 1 of [4].

Corollary 3. *Suppose that* (2.4) *is satisfied and that, for all* $\alpha \in (0,1)$,

$$V'_{(f)}(t,x) \le \alpha^{-1}\psi(t) + \varepsilon(\alpha)\{\varphi(t)+k(t)\} - W(\hat{V})k(t)$$

for all $(t,x) \in I \times B$, *where* φ *and* k *satisfy* (2.6) *and* (2.11), *where* $\psi \in L^1(I)$, *and* ε *is a non–negative continuous function on* $(0,1)$, *with* $\varepsilon(\alpha) \to 0$ *as* $\alpha \to 0$.

Let $x = x(t)$ *be a solution of* (2.1) *which ultimately lies in* B *and also satisfies* (2.8). *Then the conclusion of Theorem 1 continues to hold.*

Proof. As in the proof of Theorem 1, we suppose for contradiction that there exist $\ell > 0$ and T_1 so large that

$$\hat{V}(t,x(t)) \ge \ell \qquad \text{for all } t \ge T_1.$$

Let $\overline{m}$ denote the left hand side of (2.11), and choose α so that

$$\varepsilon(\alpha) \le \frac{W(\ell)}{2(m+1)}, \qquad m = \max\{0, \overline{m}\}.$$

Then one finds without difficulty that

$$\frac{dV}{dt}(t,x(t)) \le \tilde{\varphi}(t) - \tfrac{1}{2}W(\ell)k(t), \qquad t \ge T_1,$$

where $\tilde{\varphi}(t) = \alpha^{-1}\psi(t) + \varepsilon(\alpha)[\varphi(t) - m\,k(t)]$, $t \in I$. Moreover, since $\psi \in L^1(I)$ and $k \notin L^1(I)$, we have

$$\liminf_{t\to\infty} \int_T^t \tilde{\varphi}(s)ds \Bigg/ \int_T^t k(s)ds = \varepsilon(\alpha)\left[\liminf_{t\to\infty} \int_T^t \varphi(s)ds \Bigg/ \int_T^t k(s)ds - m\right] \le 0. \qquad \blacksquare$$

The proof of Theorem 1 now applies, mutatis mutandis, see (2.9), yielding the required contradiction.

Theorem 2. *Let the hypotheses of Theorem 1 hold, except that* (2.7) *is replaced by*

$$\liminf_{t\to\infty} \int_T^t \varphi(s)\exp\left(-\int_t^s k(r)dr\right)ds \le 0, \tag{2.12}$$

and (2.8) *by*

$$\lim_{t\to\infty} V(t,x(t)) = 0. \tag{2.13}$$

Then the conclusion of Theorem 1 remains valid.

Remark. As shown in [4], condition (2.12) is implied by (2.7). Thus when (2.13) holds the result of Theorem 2 is stronger than that of Theorem 1. Condition (2.12) first appears in [7].

Proof. As in the proof of Theorem 1, we suppose for contradiction that there exist $\ell > 0$ and T_1 so large that $\hat{V}(t, x(t)) \geq \ell$ for all $t \geq T_1$. Put

$$\omega(t) = \exp \int_T^t k(s)ds \qquad \Big(\to \infty \quad \text{as } t \to \infty\Big).$$

From (2.5) there results

$$\frac{d}{dt}\,\omega(t)V(t,x(t)) \leq \omega(t)\{\varphi(t) - W(\hat{V}(t,x(t))k(t) + V(t,x(t))k(t)\}.$$

Then, for $T_2 \geq T_1$ sufficiently large, we find from (2.13) that

$$\frac{d}{dt}\,\omega(t)V(t,x(t)) \leq \omega(t)\{\varphi(t) - \tfrac{1}{2}W(\ell)k(t)\} \qquad \text{for } t \geq T_2,$$

as in (2.9). Integrating from T to t we get, since φ, $k \in L^1_{\text{loc}}(I)$,

$$\omega(t)\,V(t,x(t)) \leq \int_T^t \varphi(s)\omega(s)ds - \tfrac{1}{2}W(\ell)\int_T^t k(s)\omega(s)ds + \text{Const.}$$

On the other hand, $\int_T^t k(s)\omega(s)ds = \int_T^t \omega'(s)ds = \omega(t) - \omega(T)$, so that

$$V(t,x(t)) \leq \frac{1}{\omega(t)}\int_T^t \varphi(s)\omega(s)ds - \tfrac{1}{2}W(\ell)\left\{1 - \frac{\text{Const.}}{\omega(t)}\right\}.$$

Finally, using (2.12) and (2.13), we obtain

$$0 = \liminf_{t\to\infty} V(t,x(t)) \leq -\tfrac{1}{2}W(\ell),$$

which is impossible.

With the additional condition (2.13), Corollaries 1–3 have obvious extensions to the case where (2.12) holds instead of (2.7).

3. Second set of results. The theorems of this section extend well-known results of Marachkov [5], see also [1, Theorems 1.1.3 and 1.1.5], and Salvadori [9], see [1, Theorem 1.1.4]. They also include Theorems 1 and 2 of Yoshizawa [10].

We begin with the most general statement of our result and follow this with a number of corollary remarks.

Theorem 3. *Suppose that* (2.4) *holds and* V *satisfies*

$$V'_{(f)}(t,x) \leq -W(\hat{V})k(t) + \psi(t) \tag{3.1}$$

for $(t,x) \in I \times B$, *where*

$$k \geq 0, \qquad k \in L^1_{\text{loc}}(I), \qquad \psi \in L^1(I). \tag{3.2}$$

Assume also that there exists $\eta > 0$ *such that*

$$\hat{V}'_{(f)}(t,x) \leq \hat{W}(1/\hat{V})\hat{k}(t) + \hat{\psi}(t) \tag{3.3}$$

for all $t \in I$ *and* $x \in \mathbb{R}^N$ *satisfying* $\hat{V}(t,x) \leq \eta$, *where*

$$\hat{k} \geq 0, \qquad \hat{k} \in L^1_{\text{loc}}(I), \tag{3.4}$$

$$\int_J k(s)\,ds \geq \tilde{W}\left(\int_J [1 + \hat{k}(s)]\,ds\right) \tag{3.5}$$

$$\hat{\psi} \in L^1_{\text{loc}}(I), \qquad \int_J \hat{\psi}(s)\,ds \leq \tilde{\tilde{W}}(|J|), \tag{3.6}$$

for all intervals $J \subset I$ *with* $|J|$ *suitably small.*

Let $x = x(t)$ *be a solution of* (2.1) *which ultimately lies in* B *and satisfies* (2.8). *Then*

$$\lim_{t\to\infty} \hat{V}(t, x(t)) = 0.$$

If (2.4) is strengthened to (2.4)$'$, then the conclusion can be improved to the canonical result $\lim_{t\to\infty} x(t) = 0$.

Proof of Theorem 3. Assume for contradiction that the conclusion fails. Then from (2.4) there exists $\ell > 0$ and a sequence $t_n \nearrow \infty$ such that

$$\hat{V}(t_n,\, x(t_n)) \geq \ell \qquad \text{for all } n.$$

We can assume of course that $\ell \leq \eta$. Now fix $\beta > 0$ so that $\tilde{\tilde{W}}(\beta) \leq \ell/4$ and also such that (3.5) and (3.6) hold whenever $|J| \leq \beta$. By refinement of the sequence (t_n), if necessary, we can suppose moreover that $t_{n+1} - t_n > \beta$ for all n.

Now define $\delta_n > 0$ as follows. If

$$\int_{t_n-\beta}^{t_n} \hat{k}(s)\,ds \le \ell/4\hat{W}(2/\ell)$$

we take $\delta_n = \beta$. Otherwise we choose $\delta_n \in (0,\beta)$ so that

$$\int_{t_n-\delta_n}^{t_n} \hat{k}(s)\,ds = \ell/4\hat{W}(2/\ell).$$

Since δ_n is always less than or equal to β, we have $t_n - \delta_n > t_{n-1}$, so the intervals $[t_n - \delta_n,\ t_n]$, $n = 1, 2, \ldots$, are disjoint.

We claim that

$$\hat{V}(t,\, x(t)) \ge \ell/2 \qquad \text{for } t \in [t_n - \delta_n,\, t_n], \quad n \ge 1. \tag{3.7}$$

Indeed let τ_n be the smallest $\tau \in [t_n - \delta_n,\ t_n)$ such that (3.7) holds in $[\tau, t_n]$. It now follows by (3.3) that, for $t \in [\tau_n, t_n]$,

$$\begin{aligned}\hat{V}(t, x(t)) &\ge \ell - \int_t^{t_n} \hat{W}(2/\ell)\hat{k}(s)\,ds - \int_t^{t_n} \hat{\psi}(s)\,ds \\ &> \ell - \ell/4 - \ell/4 = \ell/2,\end{aligned}$$

by (3.6) and the choice of β. Therefore $\tau_n = t_n - \delta_n$ and the claim is proved.

Next, from (3.1) and (3.7), for all sufficiently large t,

$$\frac{dV}{dt}(t, x(t)) \le \begin{cases} -W(\ell/2)k(t) + \psi(t), & t \in [t_n - \delta_n, t_n], \\ \psi(t), & \text{otherwise.} \end{cases}$$

Let the right hand side of this inequality be denoted by $-p(t)$, $t \in I$. Hence by (2.8) and the lemma,

$$\limsup_{t\to\infty} \int_T^t p(s)\,ds < \infty. \tag{3.8}$$

On the other hand,

$$\begin{aligned}\int_T^{t_n} p(s)\,ds &\ge -\int_T^{t_n} \psi(s)\,ds + W(\ell/2)\sum_{j=1}^{n} \int_{t_j-\delta_j}^{t_j} k(s)\,ds \\ &\ge -\int_T^{t_n} \psi(s)\,ds + W(\ell/2)\sum_{j=1}^{n} \tilde{W}\left(\int_{t_j-\delta_j}^{t_j} [1+\hat{k}(s)]\,ds\right)\end{aligned}$$

by (3.5). Moreover

$$\int_{t_n-\delta_n}^{t_n} [1+\hat{k}(s)]\,ds \geq \begin{cases} \beta & \text{if } \delta_n = \beta \\ \ell/4\hat{W}(2/\ell) & \text{if } \delta_n < \beta. \end{cases}$$

Let $\sigma(>0)$ be the minima of the two numbers on the right hand side. Then

$$\int_T^{t_n} p(s)\,ds \geq -\int_T^{t_n} \psi(s)\,ds + n\,W(\ell/2)\tilde{W}(\sigma) \to \infty \qquad \text{as } n \to \infty,$$

since $\psi \in L^1(I)$. This contradicts (3.8) and completes the proof.

It is worth noting that condition (3.3) in this theorem can be replaced by the alternative assumption

$$-\hat{V}'_{(f)}(t,x) \leq \hat{W}(1/\hat{V})\hat{k}(t) + \hat{\psi}(t)$$

without significantly altering the proof (i.e. the intervals $[t_n - \delta_n,\, t_n]$ are then replaced by $[t_n,\, t_n + \delta_n]$). A similar observation first appeared in [3, Theorem 1(b)].

This remark of course applies equally in what follows.

As indicated at the beginning of this section, Theorem 3 contains a number of special cases which are of independent interest, and can be considered direct generalizations of earlier results in Lyapunov theory. We present these cases in a series of corollaries.

Corollary 1. *Suppose that*

$$V'_{(f)}(t,x) \leq -W(|x|)k(t) + \psi(t) \tag{3.9}$$

for all $t \in I$ and $x \in B$, where k and ψ satisfy (3.2). *Assume also that*

$$x \cdot f(t,x) \leq \hat{W}(1/|x|)\hat{k}(t) + \hat{\psi}(t) \tag{3.10}$$

for all $t \in I$ and $|x| \leq \eta$, $\eta > 0$, where $\hat{k}$ and $\hat{\psi}$ satisfy (3.4)–(3.6).

Then every solution $x = x(t)$ of (2.1), *which ultimately lies in B and satisfies* (2.8), *approaches zero as $t \to \infty$.*

Proof. We choose $\hat{V}(t,x) = |x|^2$. Then $\hat{V}$ is of class C^1, and $(2.4)'$ obviously holds with $W_1(s) = s^2$. Similarly (3.1) is satisfied with a corresponding identification of wedges.

Moreover for $\hat{V}(t,x) \leq \eta^2$ we have, using (3.10),

$$\hat{V}'_{(f)}(t,x) = 2x \cdot f(t,x) \leq 2\{\hat{W}(\hat{V}^{-1/2})\hat{k}(t) + \hat{\psi}(t)\}.$$

Thus (3.3) holds, with the obvious identifications of the functions involved. The conclusion now follows at once from Theorem 3 and $(2.4)'$.

As noted in the remark above, Corollary 1 continues to hold when $x \cdot f(t,x)$ in (3.10) is replaced by $-x \cdot f(t,x)$.

Corollary 2. *Suppose that* (3.2), (3.6) *and* (3.9) *are satisfied. Assume that*

$$x \cdot f(t,x) \le \hat{\psi}(t) \quad (\text{or } \ -x \cdot f(t,x) \le \hat{\psi}(t)) \quad \textit{for } t \in I \textit{ and } |x| \le \eta, \tag{3.11}$$

and also

$$\int_E k(s)\, ds \ge \tilde{W}(|E|) \tag{3.12}$$

for all (bounded) intervals $E \subset I$. Then every solution $x = x(t)$ of (2.1)*, which lies in B and satisfies* (2.8)*, approaches zero as $t \to \infty$.*

Proof. In Corollary 1 we take $\hat{k} \equiv 0$. Then (3.10) reduces to (3.11) and (3.5) reduces to (3.12). Thus Corollary 2 follows from Corollary 1.

Marachkov's theorem [5] is the special case of Corollary 2 when $k \equiv 1$, $\psi \equiv 0$ in (3.9), and (3.11) is replaced by $|f(t,x)| \le M$, where M is a non-negative constant. Note that (3.2), (3.6), and (3.12) are then automatically satisfied.

The next corollary is a generalization of results of Lasalle [3] and Salvadori [9].

Corollary 3. *Assume that* (2.4), (3.1), (3.2), (3.6) *and* (3.12) *are satisfied. Suppose also that*

$$\hat{V}'_{(f)}(t,x) \le \hat{\psi}(t) \qquad (\textit{or } \ -\hat{V}'_{(f)}(t,x) \le \hat{\psi}(t)) \qquad \textit{when } \hat{V}(t,x) \le \eta. \tag{3.13}$$

Let $x = x(t)$ be a solution of (2.1) *which ultimately lies in B and satisfies* (2.8)*. Then the conclusion of Theorem 3 holds.*

The proof is essentially the same as that of Corollary 2, that is, take $\hat{k} \equiv 0$ in Theorem 3.

Remarks.

1. Obviously Corollary 3 reduces to Corollary 2 when $\hat{V}(t,x) = |x|^2$.
2. Salvadori's theorem [1, Theorem 1.1.4] is the special case of Corollary 3 with $k \equiv 1$, $\psi \equiv 0$, $\hat{\psi} \equiv M$ and $V(t,x) \ge 0$; note that (2.8), (3.2), (3.6) and (3.12) are then automatically satisfied.
3. Corollary 3 is almost exactly Theorem 1 of [10], with several minor exceptions. First it is assumed here that $\psi \in L^1(I)$,

while in [10] a weaker condition is stated for ψ. It seems to the authors, however, that this weaker condition is inaccurate and leads to a gap in the proof (see [10], page 1160) because γ need not be in $L^1[t_1,\infty)$. Second, Yoshizawa replaces (3.13) by the abstract condition (d), whose only purpose is to obtain directly the claim (3.7), without the help of (3.13). Finally, Yoshizawa formulates his result with respect to the stability of a general set E, as indicated in Section 4 below, rather than with respect to the behavior of $\hat{V}(t,x(t))$ as $t\to\infty$. Thus his conclusion becomes exactly (4.2) of the following section.

Corollary 4. *Suppose that $\hat{k}$ in Theorem 3 satisfies the additional condition*

$$\hat{k}(t)\le a+bk(t), \tag{3.14}$$

where a, b are non-negative constants. Then Theorem 3 remains valid with (3.5) *replaced by* (3.12).

Proof. It is enough to construct a new wedge for which (3.5) is satisfied, call it Z. Indeed we claim that

$$Z(s)=\min\left\{\frac{s}{1+a+b},\quad \tilde{W}\left(\frac{s}{1+a+b}\right)\right\}$$

suffices, where $\tilde{W}$ is the wedge in (3.12). This is easily checked, namely

$$\begin{aligned}
Z\left(\int_E(1+\hat{k})\,ds\right) &\le Z\left((1+a)|E|+b\int_E k\,ds\right)\\
&\le \begin{cases} Z((1+a+b)|E|) & \text{if } \int_E k\,ds\le |E|\\ Z\left((1+a+b)\int_E k\,ds\right) & \text{if } \int_E k\,ds\ge |E|\end{cases}\\
&\le \begin{cases} \tilde{W}(|E|) \\ \int_E k\,ds \end{cases} \qquad \text{by construction of } Z\\
&\le \int_E k\,ds, \qquad \text{by (3.12).}
\end{aligned}$$

Corollary 5. *Assume that* (2.4), (3.1)–(3.4) *hold with*

$$\hat{\psi}\ge 0,\qquad \hat{\psi}\in L^1(I),\qquad \hat{k}\notin L^1(I). \tag{3.15}$$

Suppose also that

$$\int_E k(s)\,ds \geq \tilde{W}\left(\int_E \hat{k}(s)\,ds\right) \tag{3.16}$$

for all (bounded) intervals $E \subset I$.

Let $x = x(t)$ *be a solution of* (2.1) *which ultimately lies in* B *and satisfies* (2.8)*. Then the conclusion of Theorem 3 remains valid.*

Proof. This is the same as for Theorem 3 except we now take the sequence (t_n) to satisfy (instead of $t_{n+1} - t_n > \beta$) the conditions

$$\int_{t_n}^{t_{n+1}} \hat{k}(s)\,ds > \ell/4\hat{W}(2/\ell), \qquad \int_{t_n}^{t_{n+1}} \hat{\psi}(s)\,ds < \ell/4;$$

this can always be done, since $\hat{k} \notin L^1(I)$ and $\hat{\psi} \in L^1(I)$. The corresponding sequence (δ_n) can then be chosen so that, in all cases,

$$\int_{t_{n-\delta_n}}^{t_n} \hat{k}(s)\,ds = \ell/4\hat{W}(2/\ell).$$

Clearly $t_n - \delta_n > t_{n-1}$. The rest of the proof is now the same as before, once we note from (3.16) that

$$\int_{t_n-\delta_n}^{t_n} k(s)\,ds \geq \tilde{W}\left(\int_{t_n-\delta_n}^{t_n} \hat{k}(s)\,ds\right) = \tilde{W}(\sigma),$$

where $\sigma = \ell/4\hat{W}(2/\ell)$.

Corollary 6. *Assume that* (2.4), (3.1)–(3.3) *and* (3.15) *hold, and that* $\hat{k} = k$*. Then every solution* $x = x(t)$ *of* (2.1)*, which ultimately lies in* B *and satisfies* (2.8)*, approaches zero as* $t \to \infty$.

Proof. Since $\hat{k} = k$, conditions (3.4) and (3.16) automatically hold, with $\tilde{W}(s) = s$. Hence Corollary 5 can be applied.

Corollary 6 is essentially the same as Theorem 2 in [10]. Note that the difference between Corollary 4 and Corollary 6 is the stronger condition (3.15) instead of (3.6), and the fact that condition (3.12) is then unnecessary.

4. General attracting sets. The remark following the statement of Theorems 1 and 3 allows a simple extension to the case when the attractor $x = 0$ is replaced by a general closed set E. In particular suppose $(2.4)'$ is modified to the weaker form

$$W_1(d) \le \hat{V}(t, x), \qquad d = \operatorname{dist}(x, E), \quad (t, x) \in I \times B, \tag{4.1}$$

where E is a given closed subset of B. Then, without change in the analysis, the conclusion $\lim_{t\to\infty} x(t) = 0$ of the remark is replaced by

$$\operatorname{dist}(x(t), E) \to 0 \quad \text{as} \quad t \to \infty. \tag{4.2}$$

Since $(2.4)'$ is not used in Corollaries 1 and 2 of Theorem 3, it is important to note as well that the appropriate modification of these corollaries consists, not in assuming (4.1), but rather in replacing $W(|x|)$ and $\hat{W}(1/|x|)$ in (3.9)–(3.10) by $W(d)$ and $\hat{W}(1/d)$, and $x \cdot f$ in (3.10)–(3.11) by $d_x \cdot f$. With this modification, the appropriate function $\hat{V}$ in the proofs of Corollaries 1–2 is

$$\hat{V}(t, x) = \operatorname{dist}(x, E).$$

Although $\hat{V}$ is not of class C^1, it is Lipschitz continuous. Hence $\hat{V}(t, x(t))$ is absolutely continuous on I along solutions of (2.1), and the proofs carry over without change.

A theorem of Matrosov [6], see also [1, Theorem 1.4.3], is closely connected with this extension. As presented in [1], condition (ii) of Theorem 1.4.3 is a stronger version of the hypothesis (3.9) of Corollary 2 of Theorem 3, with $W(|x|)$ replaced by $W(d)$. Hence (ii) serves the purpose of obtaining (4.2). Condition (iii) in the theorem then gives a separate Lyapunov inequality[(1)] which shows further that if (4.2) holds then $x(t) \to 0$ as $t \to \infty$.

While this result is itself not of great interest, its argument allows the following obvious generalization.

Theorem 4. *Let (E_i) be a decreasing sequence of closed sets with property that*

$$x(t) \in E_i \quad \textit{for all} \quad t \ge T_i \quad \Rightarrow \quad x(t) \in E_{i+1} \quad \textit{for all} \quad t \ge T_{i+1} \tag{4.3}$$

[(1)] If one makes the substitution $V_3(t, x) = b - V_2(t, x)$, where $b = \sup\{V_2(t, x) : (t, x) \in \mathbb{R}_+ \times S(\rho)\}$, then condition (iii) becomes $D^+V_3(t, x) \le -\xi$, which is essentially condition (3.9) of Corollary 2 again.

for any solution $x = x(t)$ *of* (2.1). *Assume also*

$$\bigcap_{1}^{\infty} E_i = \{0\}. \tag{4.4}$$

Then every solution of (2.1) *which ultimately lies in* E_1 *approaches zero as* $t \to \infty$.

In order to apply this result one must of course verify condition (4.3). As in Matrosov's theorem this can be done by using, for each i, an appropriate Lyapunov inequality. Theorems 1–3 and their corollaries, particularly in the case when $(2.4)'$ is replaced by an inequality of type (4.1), clearly can serve for this purpose.

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