# Existence of radial ground states for quasilinear elliptic equations

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**Abstract.** Existence of nontrivial nonnegative radial solutions of quasilinear equations $-\mathrm{div}(A(|\nabla u|)\nabla u) = f(u)$ in $\mathbf{R}^n$ is proved under general assumptions on the nonlinearity $f$ and the function $A$, without requiring homogeinity.

## 1 Introduction

Recently Gazzola, Serrin and Tang in [9] under general conditions on the forcing nonlinearity term $f$ proved existence of nontrivial radial ground states in $\mathbf{R}^n$ for the $m$-Laplacian equation when $m > 1$. The main purpose of this paper is to extend these existence results to general quasilinear elliptic problems, using the same separation arguments based on a variational identity technique started in [12] and [6]. In this way we are also able to cover problems in which the divergence part is given, for instance by $\mathrm{div}(Du + \varepsilon|Du|^4 Du)$, $\varepsilon > 0$, proposed by Benci, Fortunato and Pisani in a series of papers see [2] to obtain existence of static solutions in dimension 3 of certain semilinear hyperbolic generalized sine-Gordon equations, useful in many physical applications.

In particular, we are interested in finding sufficient conditions for existence of radial ground states of the quasilinear elliptic equation

$$-\mathrm{div}(A(|\nabla u|)\nabla u) = f(u) \quad \text{in } \mathbf{R}^n,\ n > 1, \tag{1.1}$$

and by a ground state we mean a nonnegative nontrivial solution of (1.1) which tends to zero at infinity.

In addition to the ground state problem (1.1), we can also consider existence of nontrivial radial solutions of the homogeneous Dirichlet-Neumann free boundary problem

$$\begin{cases} -\mathrm{div}(A(|\nabla u|)\nabla u) = f(u), & \text{in } B(0,R), \\ u \geq 0 \text{ in } B(0,R), & u \equiv \dfrac{\partial u}{\partial n} \equiv 0 \text{ on } \partial B(0,R). \end{cases} \tag{1.2}$$

A radial solution of (1.1), or (1.2), of course can be considered as a solution of the equation

$$-[r^{n-1}A(|u'(r)|)u'(r)]' = r^{n-1}f(u(r)), \quad 0 < r = |x| \le R \le \infty. \qquad (1.3)$$

For $t > 0$ we set $\Omega(t) = tA(t)$ and assume that

($\Omega 1$) *$\Omega$ is of class $C^1(0,\infty)$.*

($\Omega 2$) *$\Omega'(t) > 0$ for $t > 0$, and $\Omega(t) \to 0$ as $t \to 0$.*

($\Omega 3$) *There exists a positive number $m > 1$ such that $t^{1-m}\Omega(t)$ is a nondecreasing function on $(0,\infty)$.*

We define for $t > 0$

$$G(t) = \int_0^t \Omega(s)\,\mathrm{d}s, \quad H(t) = t\Omega(t) - G(t) = \int_0^t s\Omega'(s)\,\mathrm{d}s, \qquad (1.4)$$

so that $G$ is strictly convex and $H$ is strictly increasing and positive by ($\Omega 2$).
Note that ($\Omega 3$) is equivalent to

$$\Omega'(t) \ge \frac{m-1}{t}\Omega(t), \quad t > 0,$$

which is the property obtained in the lemma of the introduction of [13] for $m > 1$, under the main assumption for the uniqueness problem, treated in [13], that $t\Omega(t)/G(t)$ is nondecreasing on $(0,\infty)$.

As in [9], and more generally in natural existence settings, we are concerned with subcritical nonlinearities $f$ when $1 < m < n$, and possibly with exponential growth at infinity when $m = n$. The specific requests are given precisely in the statements of the main Theorems 3.1 and 4.1 below. Throughout the paper $f$ is assumed of the type ($f1$) and ($f2$), as in [9], [13] and [14]:

($f1$) $f$ is locally Lipschitz continuous on $(0,\infty)$ and $\int_0^1 |f(u)|\,\mathrm{d}u < \infty$.

By ($f1$) it is clear that $F(u) = \int_0^u f(v)\,\mathrm{d}v$ is well defined and is continuous on $[0,\infty)$, with $F(0) = 0$.

($f2$) There exists $\beta > 0$ such that $F(u) < 0$ for $0 < u < \beta$, $F(\beta) = 0$ and $f(\beta) > 0$.

The behavior of $f$ near zero is of crucial importance for the existence results given in Sections 3 and 4. In the following considerations we shall identify two mutually exclusive situations:

1. *Regular case*: $f$ can be extended by continuity at $u = 0$, with $f(0) \le 0$ by ($f2$).

2. *Singular case*: $\liminf_{u\to 0^+} f(u) < \limsup_{u\to 0^+} f(u)$ and put $f(0) = 0$, so that $f$ is discontinuous at $u = 0$.

As mentioned above, the main purpose of this paper is to extend to the general equations (1.1) and (1.2) the recent existence results given by Gazzola, Serrin and Tang in [9] for such nonlinearities $f$ in the $m$-Laplacian case, namely when $\Omega(t) = t^{m-1}$, $m > 1$, in ($\Omega$1)-($\Omega$3).

For a complete discussion on the wide background and literature concerning related previous results, also for the classical scalar field equation, we recall [9] and the references therein.

We also observe in passing that the main existence Theorems 3.1 and 4.1 given below partially answer a conjecture given in [13], Section 6.1. Indeed, when $m > 1$ and $t\Omega(t)/G(t)$ is nondecreasing on $(0,\infty)$, which is the main condition required for uniqueness in [13], then ($\Omega$3) holds.

The paper is organized as follows: in Section 2 some preliminary qualitative properties for solutions of (1.1) and (1.2) are given, including a continuous data dependence result and an underlying variational inequality. In Section 3 the main existence Theorem 3.1 is established and proved in the case $1 < m \le n$ under the main subcritical growth restriction (3.1) on $f$. The section ends with many remarks on recent related results as uniqueness, symmetry and different qualitative properties for compactly or noncompactly supported ground states, and with a corollary of independent interest. Section 4 is devoted to the extension (Theorem 4.1) of the previous existence criterion also to the case $n < m$ under a further restriction on $\Omega$, which is of course automatic in the $m$–Laplacian case, as well as in many divergence operators arising in applications (see, i.e., [2]). Finally, in Section 5 some $L^\infty$ a priori estimates are deduced for solutions of (1.1) and (1.2), in terms of the main parameters $m$, $n$, and of the main functions $f$ and $\Omega$. These estimates are also numerically presented for concrete equations and nonlinearities when either $n < m$ or $m \le n$. The paper ends with Theorem 5.2 which provides existence when $n = m$ and $f$ could have exponential growth at infinity.

# 2 Preliminary results for existence

We shall treat *classical* solutions of (1.3), namely functions $u$ of class $C^1$ on $(0, R)$ such that $u'(0) = 0$ and $A(|u'|)u' \in C^1$. Hence classical solutions of (1.3) satisfy the initial value problem

$$\begin{cases} -[r^{n-1}A(|u'(r)|)u'(r)]' = r^{n-1}f(u(r)), \quad 0 < r < R \\ u(0) = \alpha > 0, \; u'(0) = 0, \end{cases} \tag{2.1}$$

for some $\alpha > 0$.

As proved in Lemma 1.1.1 in [6], if $u$ is a local classical solution of (2.1), then $A(|u'|)u'$ is of class $C^1$ also at $r = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[A(u'(r))u'(r)\right]\Big|_{r=0} = \lim_{r \longrightarrow 0^+} \frac{A(u'(r))u'(r)}{r}.$$

Indeed, (2.1) implies

$$\frac{A(u'(r))u'(r)}{r} = -\frac{1}{r^n}\int_0^r s^{n-1} f(u(s))\,\mathrm{d}s,$$

and, by De L'Hopital rule,

$$\lim_{r \longrightarrow 0^+} \frac{1}{r^n}\int_0^r s^{n-1} f(u(s))\,\mathrm{d}s = \frac{f(\alpha)}{n},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[A(u'(r))u'(r)\right]\Big|_{r=0} = -\frac{f(\alpha)}{n}. \tag{2.2}$$

Consequently, as shown in Lemma 1.2.1 of [6], the energy function

$$E(r) = H(u'(r)) + F(u(r)) \tag{2.3}$$

is differentiable along any classical solution of (2.1), with

$$E'(r) = -\frac{n-1}{r}|u'(r)|^2 A(|u'(r)|) = -\frac{n-1}{r}\rho(r)\Omega(\rho(r)), \tag{2.4}$$

where $\rho(r) = |u'(r)|$. Hence $E$ is nonincreasing and bounded below on $(0, R)$, since $F(u(r))$ is bounded below by $(f2)$ and $H(u'(r)) \geq 0$ by (1.4). Of course $E(0) = F(\alpha)$.

If $R = \infty$ and $\lim_{r\to\infty} u(r) = 0$, then $\lim_{r\to\infty} F(u(r)) = 0$ by $(f1)$, and there exists finite $\lim_{r\to\infty} E(r)$ by the arguments above. From

$$E(r) = F(\alpha) - (n-1)\int_0^r \frac{\rho\Omega(\rho)}{s}\,\mathrm{d}s,$$

the integral approaches a finite limit as $r \to \infty$, and in turn $H(u'(r))$ tends to a finite limit as $r \to \infty$ by (2.3). Since $H$ is strictly increasing by (1.4), this implies that also $u'(r)$ tends to a limit as $r \to \infty$, and so $\lim_{r\to\infty} u'(r) = 0$.

Therefore a classical radial ground state of (1.1), or a classical solution of the free boundary problem (1.2), has the property

$$u(0) = \alpha, \; u'(0) = 0, \; u(R) = u'(R) = 0,$$

with unified notation in both cases $R = \infty$, or $R < \infty$. Hence by (2.4) for $0 \leq r < R$

$$(n-1) \int_r^R \frac{\rho\Omega(\rho)}{s} \, \mathrm{d}s = H(u'(r)) + F(u(r)) > 0,$$

when $u$ is a nontrivial solution, and in particular

$$F(\alpha) > 0,$$

in other words for nontrivial classical solutions of (1.1) or (1.2) it holds

$$u(0) = \alpha > \beta,$$

by $(f2)$.

Consequently, from now on, we shall assume in problem (2.1) that

$$\alpha \geq \beta, \tag{2.5}$$

since the case $u(0) = \beta$, which does not occur for the solution searched, is of interest in later work. In the paper the superscript $'$ will denote differentiation depending on the context. This will not cause confusion.

**Lemma 2.1** *Assume (2.5). Problem (2.1) has a unique classical solution $u$ in a neighborhood of the origin. Moreover, $u'(r) < 0$ for $r$ small and positive.*

**Proof.** Local existence and uniqueness of $C^1$ solution of the Cauchy problem (2.1) follows from Proposition $A4$ of the Appendix of [6], since ($\Omega 3$) is equivalent to

$$\Omega'(t) \geq \frac{m-1}{t} \Omega(t), \tag{2.6}$$

and in turn for $0 < t < m-1$

$$\Omega'(t) \geq \Omega(t) > 0, \tag{2.7}$$

which is the main growth condition, in the special case $\mu = 1$, required. By (2.5) and $(f2)$ we have $f(\alpha) > 0$, so that $u'(r)$ is negative for small $r$ by (2.2) and ($\Omega 2$). ∎

By Lemma 2.1 it follows that $u$ remains a classical solution of (2.8) in the maximal domain of continuation where it is positive.

We denote by $J = (0, R)$, $R$ finite or not, the maximal open interval of continuation under the restriction

$$u > 0, \quad -\infty < u' < 0 \quad \text{in } J.$$

It is clear that, since $0 < u < \alpha$ in J, the continuation and the corresponding interval $J$ is uniquely determined. In the sequel we understand that every solution $u$ of (2.1) is continued exactly to the corresponding maximal domain $J = (0, R)$.

Since $u$ is decreasing and positive in $J$, it is obvious that $\lim_{r\to R^-} u(r)$ exists and is nonnegative. We denote this limit by $\ell$.

The definition of $J$ shows that we deal with solution of (2.1) such that $u'(r) < 0$. Setting $\rho = \rho(r) = |u'(r)|$ as before, problem (2.1) can then be written as

$$\begin{cases} [r^{n-1}\Omega(\rho(r))]' - r^{n-1}f(u(r)) = 0, \quad 0 < r < R, \\ u(0) = \alpha > 0, \ u'(0) = 0, \end{cases} \tag{2.8}$$

or equivalently

$$\begin{cases} [\Omega(\rho(r))]' + \dfrac{n-1}{r}\Omega(\rho(r)) - f(u(r)) = 0, \quad 0 < r < R, \\ u(0) = \alpha > 0, \ u'(0) = 0. \end{cases} \tag{2.9}$$

**Lemma 2.2** *Let $u$ be a solution of (2.1) and $J = (0, R)$ be the corresponding maximal interval of definition in the sense above and assume (2.5).*

*(i) If $R = \infty$ then $\lim_{r\to\infty} u'(r) = 0$.*

*(ii) If $R < \infty$ then either $u(r) \to 0$ or $u'(r) \to 0$ as $r \to R^-$. In the first case $u'(R) = \lim_{r\to R^-} u'(r)$ exists and is nonpositive.*

*(iii) The limit $\ell$ belongs to $[0, \beta)$.*

*(iv)* $\displaystyle\lim_{r\longrightarrow 0^+} \frac{\Omega(\rho(r))}{r} = \frac{f(\alpha)}{n}.$

*(v) Let $\lambda > \beta$. If $\alpha > \lambda$, then there exists a unique value $\overline{R} = \overline{R}(\alpha) \in J$ such that $u(\overline{R}) = \lambda$.*

**Proof.** **(i).** See the discussion after (2.4).

**(ii).** Since $u'$ is negative and bounded on $J$ and $R$ is assumed finite, the only obstacle to continuation to $R$ is for either $u$ or $u'$ to approach zero as $r \to R^-$, this proves the first part of the claim. Moreover, since $E$ approaches a limit as $r$ tends to $R$ by (2.4), it is clear that also $u'$ approaches a limit, but $u'$ is negative in $J$ and the claim follows.

**(iii).** By contradiction, suppose $\ell \geq \beta$. Then $\beta \leq \ell < u < \alpha$ in $J$, and this implies $[r^{n-1}\Omega(\rho(r))]' > 0$ by (2.8) and $(f2)$, that is $r^{n-1}\Omega(\rho(r))$ is strictly increasing on $J$.

If $R$ is finite and $\ell \geq \beta > 0$, point *(ii)* of this lemma gives $u'(R) = 0$, and in turn $r^{n-1}\Omega(\rho(r))$ tends to 0 as $r \to R^-$. On the other hand, while $r^{n-1}\Omega(\rho(r))$ is 0 at $r = 0$. This is impossibile since $r^{n-1}\Omega(\rho(r))$ is strictly increasing on $J$.

If $R = \infty$, by (2.9) and *(i)*, we get $\lim_{r\to\infty}[\Omega(\rho(r))]' = f(\ell) > 0$ since $\ell \geq \beta$. This is impossible since $\Omega(\rho(r)) > 0$ on $J$ and $\lim_{r\to\infty} \Omega(\rho(r)) = 0$ by $(\Omega 2)$ and *(i)*, so the claim is proved.

**(iv).** This is exactly (2.2).

**(v).** The claim follows easily since $u$ is strictly decreasing, by definition of $J$. ■

**Lemma 2.3** *Let $u_1$ be a solution of (2.8) with maximal interval $J_1$ and assume (2.5). Then for any $r_0 \in J_1$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_2$ is a solution of (2.8) with $|u_1(0) - u_2(0)| < \delta$, then $u_2(r)$ is defined on $[0, r_0]$ and*

$$\sup_{[0,r_0]} \{|u_1(r) - u_2(r)| + |u_1'(r) - u_2'(r)|\} < \varepsilon.$$

**Proof.** Let $u_1$ be a solution of the initial value problem (2.8) with $u(0) = \alpha > \beta$, and let $u_2$ be a solution of (2.1) with $u_2(0) = \alpha + h > \beta$ and $h \neq 0$. Let $J_1 = (0, R_1)$ and $J_2 = (0, R_2)$ be the corresponding maximal intervals of definition and note that in $J_1$ it holds $u_1' < 0$ and $u_2' < 0$ in $J_2$. Let $0 < \overline{r} < \min\{R_1, R_2\}$ be so small that

$$\min_{i=1,2}\{\min_{r\in[0,\overline{r}]} u_i(r)\} \geq \beta, \tag{2.10}$$

$$\max_{i=1,2}\{\max_{r\in[0,\overline{r}]} |u_i'(r)|\} \leq m - 1, \tag{2.11}$$

$$\begin{aligned} &\min\nolimits_{r\in[0,\overline{r}]} f(u_1(r)) \geq \tfrac{1}{2} f(\alpha), \\ &\min\nolimits_{r\in[0,\overline{r}]} f(u_2(r)) \geq \tfrac{2}{3} f(\alpha + h) \geq \tfrac{1}{2} f(\alpha), \end{aligned} \tag{2.12}$$

provided that also $|h| > 0$ is sufficiently small, say $0 < |h| < h_0$.

By (2.10) the function $f$ is Lipschitzian on the compact interval containing $\cup_{i=1,2} u_i([0, \overline{r}])$, and let $M$ be the Lipschitz constant. From (2.1) and (2.12) we obtain for all $0 < r < \overline{r}$ and $i = 1, 2$

$$\Omega(|\rho_i(r)|) = \int_0^r \left(\frac{s}{r}\right)^{n-1} f(u_i(s))\,\mathrm{d}s \geq \frac{r}{2n} f(\alpha), \tag{2.13}$$

provided that $0 < |h| < h_0$.

Put $\omega(r) = \Omega(\rho_1(r)) - \Omega(\rho_2(r))$. Then

$$\Omega'(p)|u_1'(r) - u_2'(r)| = |\omega(r)|, \tag{2.14}$$

where $p$ is a proper intermediate value between $\rho_1(r)$ and $\rho_2(r)$. Hence by (2.11), (2.7) and the monotonicity of $\Omega$ for all $r \in (0, \overline{r}]$, inequality (2.13) implies

$$\Omega'(p) \geq \Omega(p) \geq \min_{i=1,2}\{\Omega(\rho_i(r))\} \geq \frac{r}{2n} f(\alpha).$$

Thus (2.14) gives

$$|u_1'(r) - u_2'(r)| \leq c\frac{|\omega(r)|}{r}, \tag{2.15}$$

where $c = 2n/f(\alpha)$. By (2.8) again for $0 < r \leq \overline{r}$

$$\begin{aligned}
|\omega(r)| &\leq \int_0^r \left(\frac{s}{r}\right)^{n-1} |f(u_1(s)) - f(u_2(s))| \, \mathrm{d}s \\
&\leq \frac{r}{n} \sup_{[0,r]} |f(u_1(s)) - f(u_2(s))| \qquad (2.16) \\
&\leq \frac{Mr}{n} \left(|h| + \int_0^r |u_1'(r) - u_2'(s)| \, \mathrm{d}s\right),
\end{aligned}$$

and so

$$\frac{|\omega(r)|}{r} \leq \frac{M}{n} \left(|h| + c\int_0^r \frac{|\omega(s)|}{s} \, \mathrm{d}s\right).$$

By Gronwall's inequality applied to $|\omega(r)|/r$ we get on $[0, \overline{r}]$

$$|\omega(r)| \leq \frac{M}{n}|h|re^{Dr},$$

where $D = Mc/n$, and by (2.15)

$$|u_1'(r) - u_2'(r)| \leq D|h|e^{Dr}.$$

Therefore integration on $[0, r]$, $r \leq \overline{r}$, yields

$$|u_1(r) - u_2(r)| \leq |h|e^{D\overline{r}},$$

and the claim is proved on $[0, \overline{r}]$ provided $\overline{r}$ is sufficiently small.

Hence

$$\begin{aligned} \tilde{h} = |u_1(\overline{r}) - u_2(\overline{r})| &\leq |h|e^{D\overline{r}}, \\ |u_1'(\overline{r}) - u_2'(\overline{r})| &\leq D|h|e^{D\overline{r}}, \end{aligned}$$

for some $\overline{r} > 0$, moreover $u_1$ and $u_2$ solve (2.8). Take now $0 < r_0 < \min\{R_1, R_2\}$. Without loss of generality we assume also $r_0 > \overline{r}$, otherwise there is nothing to prove.

By our choice of $r_0$ there exists $\delta_1$,$\delta_2 > 0$ such that $\delta_1 \leq |u_i'(r)| \leq \delta_2$ in $[\overline{r}, r_0]$ for $i = 1, 2$, so that (2.14) holds in $[\overline{r}, r_0]$ with $\Omega'(p) \neq 0$. In fact, by ($\Omega$2) we can assume that $1/B \leq \Omega'(p) \leq B$ for $p \in [\delta_1, \delta_2]$, with $B > 0$. In particular

$$|\omega(\overline{r})| \leq B|u_1'(\overline{r}) - u_2'(\overline{r})| \leq BD|h|e^{D\overline{r}}.$$

Let $r \in [\overline{r}, r_0]$, obviously it holds

$$|u_1(r) - u_2(r)| \leq \tilde{h} + \int_{\overline{r}}^{r} |u_1'(s) - u_2'(s)| \,\mathrm{d}s$$

for any $r \in [\overline{r}, r_0]$. Denote by $L$ the Lipschitz constant of $f$ on the compact interval containing $\cup_{i=1,2} u_i([\overline{r}, r_0])$. Hence by (2.14)

$$\begin{aligned} |f(u_1(r)) - f(u_2(r))| &\leq L\left(\tilde{h} + \int_{\overline{r}}^{r} |u_1'(s) - u_2'(s)| \,\mathrm{d}s\right) \\ &\leq L\left(\tilde{h} + \int_{\overline{r}}^{r} \frac{|\omega(s)|}{\Omega'(p(s))} \,\mathrm{d}s\right), \end{aligned}$$

as in (2.16) we have

$$\Big||\omega(r)| - |\omega(\overline{r})|\Big| \leq \frac{L}{n} r \left[\tilde{h} + B\int_{\overline{r}}^{r} |\omega(s)| \,\mathrm{d}s\right].$$

Consequently

$$\begin{aligned} |\omega(r)| &\leq \frac{L}{n} r_0 \left[\tilde{h} + B\int_{\overline{r}}^{r} |\omega(s)| \,\mathrm{d}s\right] + |\omega(\overline{r})| \\ &\leq |h|\frac{e^{D\overline{r}}}{n}(Lr_0 + MBc) + \frac{LB}{n} r_0 \int_{\overline{r}}^{r} |\omega(s)| \,\mathrm{d}s \\ &= K|h| + T\int_{\overline{r}}^{r} |\omega(s)| \,\mathrm{d}s, \end{aligned}$$

where $K = e^{D\overline{r}}(Lr_0 + MBc)/n$ and $T = LBr_0/n$. By Gronwall's inequality it results

$$|\omega(r)| \leq K|h|e^{T(r-\overline{r})},$$

which implies

$$|u_1'(r) - u_2'(r)| \leq BK|h|e^{T(r-\overline{r})}. \tag{2.17}$$

Integrating over $[\overline{r}, r_0]$ we find

$$\begin{aligned} |u_1(r) - u_2(r)| &\leq \frac{BK}{T}|h|[e^{T(r-\overline{r})} - 1] + |u_1(\overline{r}) - u_2(\overline{r})| \\ &\leq \frac{BK}{T}|h|e^{T(r-\overline{r})} + \tilde{h} \leq \left[e^{D\overline{r}} + \frac{BK}{T}e^{T(r-\overline{r})}\right]|h|. \end{aligned} \tag{2.18}$$

Now (2.17) and (2.18) show that, if $h$ is sufficiently small, $u_1$ and $u_2$ remain close in $C^1$ topology.

Now it remains to show that for any fixed positive number $r_0 < R_1$, namely in $J_1$, and for any solution $u_2$ of (2.8) and (2.5), with $u_2(0) = \alpha + h$ and $h$ sufficiently small, then $(0, r_0) \subset J_2$. Otherwise there exists a number $R < R_1$ and a sequence of solutions $v_n$ of (2.8) with $v_n(0) = \alpha + h_n$ and $h_n \to 0$, such that $(0, R_n)$, the maximal interval of definition of $v_n$, is contained in $(0, R)$, so that $R_n \leq R < R_1$. Moreover, up to a subsequence, we can assume that $R_n \to R^-$. Since the functions $v_n$ are bounded in the $C^1$ topology, by Ascoli-Arzelà theorem $\|v_n - v\|_\infty \to 0$ in $[0, R) \subset J_1$. Of course $v(0) = \alpha$, and in turn $v \equiv u_1$ in any interval $[0, r]$ with $r < R$, by the unique continuation proved in Lemma 2.1 and the subsequent remarks. This is clearly a contradiction, since $u_1(R) > 0$ and $u_1'(R) < 0$, while $v$ cannot be continued over $R$. ■

**Lemma 2.4** *Let $u$ be a solution of (2.1). Set*

$$Q(v) = mnF(v) - (n-m)vf(v), \quad v \in \mathbf{R},$$

*and*

$$P(r) = (n-m)r^{n-1}u(r)u'(r)A(|u'(r)|) + mr^nE(r), \quad 0 < r < R.$$

*Then*

$$P(r) \geq \int_0^r Q(u(s))s^{n-1}\,ds. \tag{2.19}$$

**Proof.** By (2.1) and (2.4) direct calculation shows that

$$\begin{aligned} P'(r) &= r^{n-1}Q(u(r)) + nr^{n-1}[(1-m)\rho\Omega(\rho) + mH(\rho)] \\ &\geq r^{n-1}Q(u(r)), \end{aligned}$$

where the last inequality is a consequence of ($\Omega3$). Indeed by the definition of $G$ given in (1.4), (2.6) and integration by parts,

$$\begin{aligned} G(t) &= \int_0^t \Omega(s)\,\mathrm{d}s \leq \frac{1}{m-1}\int_0^t s\Omega'(s)\,\mathrm{d}s \\ &= \frac{1}{m-1}\left[t\Omega(t) - G(t)\right]. \end{aligned}$$

Hence

$$\frac{t\Omega(t)}{G(t)} \geq m, \tag{2.20}$$

and this implies by (1.4) that

$$H(t) \geq \frac{m-1}{m} t\Omega(t). \tag{2.21}$$

Since $P(0) = 0$, the claim (2.19) follows. ■

# 3 Existence results

In this section we state and prove the main existence theorem under assumptions ($f1$), ($f2$) and ($\Omega1$)-($\Omega3$). From now on we denote by $\gamma$ the number given by

$$\gamma = \min\{u > \beta : f(u) = 0\},$$

and $\gamma = \infty$ if $f(u) > 0$ for all $u \geq \beta$.

**Theorem 3.1** *Suppose that $\gamma = \infty$, $n \geq m$ and*

($H1$) *the function*

$$Q(v) = mnF(v) - (n-m)vf(v), \quad v \in \mathbf{R},$$

*is locally bounded near $v = 0$ and there exists $\lambda > \beta$ and $k \in (0,1)$ such that $Q(v) \geq 0$ for $v \geq \lambda$ and*

$$\limsup_{v\to\infty} Q(v_2)\left(\frac{v^{m-1}}{f(v_1)}\right)^{n/m} = \infty \tag{3.1}$$

*for all $v_1$ and $v_2$ in $[kv, v]$;*

$(H2)\quad \liminf_{u\to\infty} f(u) > n.$

*Then either (1.1) has a positive radial ground state, or (1.2) has a radial solution for some* $R > 0$*. Moreover, if* $r = |x|$*, the funtion* $u = u(r)$ *obeys* $u'(r) < 0$ *for all* $r > 0$ *such that* $u(r) > 0$*.*

Before the proof of Theorem 3.1 we give several lemmas.

Let $\alpha \geq \beta$ and let $u_\alpha$ be the corrisponding solution of (2.8) with maximal domain $J_\alpha = (0, R_\alpha)$, $R_\alpha \leq \infty$. Also put $\ell_\alpha = \lim_{r\to R_\alpha^-} u_\alpha(r)$, of course $\ell_\alpha$ belongs to $[0, \beta)$ by Lemma 2.2-($iii$).

Define

$$I^- = \{\alpha \geq \beta : R_\alpha < \infty,\ \ell_\alpha = 0,\ u'_\alpha(R_\alpha) < 0\},$$
$$I^+ = \{\alpha \geq \beta : \ell_\alpha > 0\}.$$

It is evident that $I^+$ and $I^-$ are disjoint. First we show that

**Lemma 3.2** *$\beta$ belongs to $I^+$.*

**Proof.** Let $u$ be a solution of (2.8) with $u(0) = \beta$, defined on the corresponding maximal domain $J = (0, R)$. By the definition of $E$ and assumption ($f2$) we have $E(0) = 0$, and, by (2.4), $E(r) < 0$ on $J$, since $E$ is strictly decreasing. Fix $r_0$ in $J$, then $F(u(r)) \leq E(r) < E(r_0)$ for $r_0 < r < R$, by (1.4) and (2.3). Thus $F(\ell) \leq E(r_0) < 0$ and so $\ell > 0$ by ($f1$). Hence $\beta$ belongs to $I^+$. ∎

**Lemma 3.3** *$I^+$ is open in $[\beta, \infty)$.*

**Proof.** Let $\alpha_* \in I^+$ and let $u_*$ and $J_*$ be the corresponding solution with its maximal domain $J_* = (0, R_*)$. Of course $0 < \ell_* < \beta$ and $u'_*(r) \to 0$ as $r \to R_*^-$, by Lemma 2.2 ($i$) and ($iii$), and in turn

$$\lim_{r\longrightarrow R_*^-} E_*(r) = F(\ell_*) < 0,$$

by (2.3) and ($f2$).

Let $r_0 \in J_*$ be such that $E_*(r_0) < 0$. If $\alpha$ is chosen sufficiently close to $\alpha_*$ and $u$ is the corresponding solution of (2.8) with initial value $\alpha$, then applying Lemma 2.3 we obtain that $(0, r_0] \subset J$, where $J$ is the maximal domain of $u$, while also $E(r_0) \leq \frac{1}{2}E_*(r_0) < 0$. As in the proof of Lemma 3.2, this implies that $\alpha$ belongs to $I^+$. ∎

Since $F(0) = 0$ and $F$ is positive in $(\beta, \infty)$, we can define

$$\overline{F} = \max_{[0,\beta]} |F(v)| \,, \tag{3.2}$$

of course this quantity is positive by $(f2)$. In the next result we characterize the initial values which belong to $I^-$ in the spirit of Lemma 2.1.1 of [6].

More precisely, we show that, fixed $\lambda > \beta$, any solution of the initial value problem (2.8), with $\alpha > \lambda$, which cross the line $u = \lambda$ sufficiently far from $r = 0$ will eventually reach the axis $u = 0$ with nonzero slope. This means that only solutions which cross the line $u = \lambda$ sufficiently fast can be candidates for being ground states.

**Lemma 3.4** *Let $\lambda > \beta$. For any $\alpha < \lambda$, let $\overline{R} = \overline{R}(\alpha)$ be the unique value of $r$ such that the solution $u$ of (2.8) reaches $\lambda$. Then $\alpha$ is in $I^-$ provided that*

$$\overline{R} \geq \frac{(n-1)\lambda}{F(\lambda)} \frac{\dfrac{m}{m-1}[\overline{F} + F(\lambda)]}{\Phi^{-1}\left(\dfrac{m}{m-1}[\overline{F} + F(\lambda)]\right)} = \overline{C}, \tag{3.3}$$

*where $\overline{C} = \overline{C}(\lambda)$ and $\Phi^{-1}$ denotes the inverse of the increasing function $\Phi(t) = t\Omega(t)$.*

**Proof.** First note that (3.3) is equivalent to the inequality

$$\overline{R} \geq \frac{(n-1)\lambda}{F(\lambda)} \Omega\left(\frac{n-1}{\overline{R}} \frac{\lambda}{F(\lambda)} \frac{m}{m-1}[\overline{F} + F(\lambda)]\right). \tag{3.4}$$

In fact it is easy to see that the function $[t - \Omega(c/t)]$ is increasing for any constant $c > 0$ by $(\Omega 2)$ and tends to infinity as $t \to \infty$, so (3.3) holds if and only if (3.4) holds, with $c = m[\overline{F} + F(\lambda)]/(m-1)$.

We follow the proof of Lemmas 3.1 of [9]. Let $\alpha > \lambda$ and let $u$ be the solution of (2.8) with $u(0) = \alpha$ defined on its maximal domain $J = (0, R)$, $R \leq \infty$. Suppose by contradiction that $\alpha \notin I^-$. It follows that $0 \leq \ell < \beta$ and $\lim_{r \to R^-} u'(r) = 0$ by Lemma 2.2 $(i)$ and $(iii)$, so we define $M = \max_{[\overline{r},R]} |u'| = |u'(R_0)|$, with $R_0 \in [\overline{R}, R)$.

Integrating (2.4) over $[R_0, R]$, we have

$$\begin{aligned} H(M) &= F(\ell) - F(u(R_0)) + (n-1) \int_{R_0}^{R} \frac{\rho(s)\Omega(\rho(s))}{s} \, \mathrm{d}s \\ &\leq \overline{F} + (n-1) \frac{\Omega(M)}{\overline{R}} \int_{\overline{R}}^{R} |u'(s)| \, \mathrm{d}s, \end{aligned}$$

since $E(R) = F(u(R)) = F(\ell) \le 0$. From the fact that $u(\overline{R}) - u(R) = \lambda - \ell \le \lambda$, since $\ell \ge 0$, we find

$$H(M) \le (n-1)\frac{\Omega(M)}{\overline{R}}\lambda + \overline{F}. \tag{3.5}$$

On the other hand, integrating over $[\overline{R}, R]$ and arguing as above, we have

$$\begin{aligned} F(\lambda) &< E(\overline{R}) = E(R) + (n-1)\int_{\overline{R}}^{R} \frac{\rho(s)\Omega(\rho(s))}{s}\,\mathrm{d}s \\ &< (n-1)\frac{\Omega(M)}{\overline{R}}\lambda. \end{aligned} \tag{3.6}$$

By (3.6) and (3.4) we get

$$\Omega(M) > \frac{\overline{R}}{n-1}\frac{F(\lambda)}{\lambda} \ge \Omega\left(\frac{n-1}{\overline{R}}\frac{\lambda}{F(\lambda)}\frac{m}{m-1}[\overline{F} + F(\lambda)]\right),$$

in other words

$$\frac{m-1}{m}M > \frac{(n-1)}{\overline{R}}\frac{\lambda}{F(\lambda)}[\overline{F} + F(\lambda)]. \tag{3.7}$$

Now from (3.5) and (3.7), we find that

$$\begin{aligned} \overline{F} &\ge H(M) - (n-1)\frac{\Omega(M)}{\overline{R}}\lambda \\ &\ge \frac{m-1}{m}M\Omega(M) - (n-1)\frac{\Omega(M)}{\overline{R}}\lambda \\ &= \Omega(M)\left[\frac{m-1}{m}M - \frac{(n-1)}{\overline{R}}\lambda\right] \\ &> \frac{\overline{R}}{n-1}\frac{F(\lambda)}{\lambda}\frac{n-1}{\overline{R}}\lambda\left[\frac{\overline{F} + F(\lambda)}{F(\lambda)} - 1\right] = \overline{F}, \end{aligned}$$

and this contradiction proves the claim. ∎

**Lemma 3.5** *If* $(H1)$ *and* $(H2)$ *hold, then* $I^-$ *is nonempty.*

**Proof.** By $(H1)$ there is $\lambda > \beta$ such that $Q(v) \ge 0$ for $v \ge \lambda$. Choose $\alpha > \lambda/k$, where $k \in (0,1)$ is the number given in $(H1)$, and let $u$ be a corresponding solution of (2.1), with maximal domain $J = (0, R)$, $R \le \infty$.

Since $\alpha > \lambda > \beta$, by Lemma 2.2 ($v$) there are in $J$ a unique $R_k$ and a unique $\overline{R} = \overline{R}(\alpha)$ such that

$$u(R_k) = k\alpha > u(\overline{R}) = \lambda,$$

so that clearly $R_k < \overline{R}$. Also define $\overline{\alpha}$ by $f(\overline{\alpha}) = \max_{[k\alpha,\alpha]} f$, so $\overline{\alpha} = \overline{\alpha}(k, \alpha)$. In fact, we can take $\overline{\alpha} = k_1\alpha$ for some $k_1 \in [k, 1]$.

Let $0 < r < R_k$ and integration of (2.8) on $[0, r]$ gives

$$r^{n-1}\Omega(\rho(r)) = \int_0^r s^{n-1} f(u(s))\,\mathrm{d}s \le \frac{r^n}{n} f(\overline{\alpha}),$$

namely

$$\Omega(\rho(r)) \le \frac{f(\overline{\alpha})}{n} r,$$

or

$$\rho(r) \le \Omega^{-1}\left(\frac{f(\overline{\alpha})}{n} R_k\right).$$

Integration on $[0, R_k]$ gives

$$\alpha(1-k) \le R_k \Omega^{-1}\left(\frac{f(\overline{\alpha})}{n} R_k\right), \tag{3.8}$$

in other words

$$\frac{f(\overline{\alpha})}{n} \ge \frac{1}{R_k}\Omega\left(\frac{\alpha(1-k)}{R_k}\right) \ge \frac{1}{R_k}\left(\frac{\alpha}{R_k}\right)^{m-1}\Omega(1-k), \tag{3.9}$$

if $\alpha/R_k \ge 1$, since by ($\Omega$3)

$$\Omega(\mu t) \ge \mu^{m-1}\Omega(t) \quad \text{for all } t \ge 0 \text{ and } \mu \ge 1. \tag{3.10}$$

Consequently, if $\alpha/R_k \ge 1$,

$$R_k \ge [\Omega(1-k)]^{1/m}\left(\frac{n\alpha^{m-1}}{f(\overline{\alpha})}\right)^{1/m}. \tag{3.11}$$

If $\alpha/R_k < 1$, we take $\alpha > 1$ even larger, if necessary, so that

$$\frac{f(\overline{\alpha})}{n} > 1, \tag{3.12}$$

and of course this is possible by $(H2)$. Now by (3.9) and (3.10), writing $\alpha/R_k = (\alpha/R_k)^{m/(m-1)}(R_k/\alpha)^{1/(m-1)}$, we have

$$\begin{aligned} \frac{f(\overline{\alpha})}{n} &\geq \frac{1}{R_k}\Omega\left(\left(\frac{\alpha}{R_k}\right)^{m/(m-1)}\left(\frac{R_k}{\alpha}\right)^{1/(m-1)}(1-k)\right) \\ &\geq \frac{1}{\alpha}\Omega\left(\frac{(1-k)\alpha}{R_k^{m/(m-1)}}\alpha^{1/(m-1)}\right) \geq \Omega\left(\frac{(1-k)\alpha}{R_k^{m/(m-1)}}\right). \end{aligned}$$

Therefore

$$R_k^{m/(m-1)} \geq \frac{(1-k)\alpha}{\Omega^{-1}\left(f(\overline{\alpha})/n\right)}.$$

By (3.10) and (3.12)

$$R_k \geq \left[\frac{(1-k)\alpha}{\Omega^{-1}(f(\overline{\alpha})/n)}\right]^{(m-1)/m} \geq \left[\frac{1-k}{\Omega^{-1}(1)}\right]^{(m-1)/m}\left(\frac{n\alpha^{m-1}}{f(\overline{\alpha})}\right)^{1/m}. \tag{3.13}$$

In conclusion, for $\alpha$ sufficiently large, by (3.11) and (3.13) it results

$$R_k \geq \left(\frac{dn\alpha^{m-1}}{f(\overline{\alpha})}\right)^{1/m}, \quad \text{where } d = \min\left\{\left[\frac{(1-k)}{\Omega^{-1}(1)}\right]^{m-1}, \Omega(1-k)\right\}. \tag{3.14}$$

Since $Q(v) \geq 0$ for $v \geq \beta$ and $Q$ is locally bounded below near $v = 0$, by $(H2)$, we can define

$$\overline{Q} = -\inf_{s>0} Q(s) < \infty.$$

Also $Q(\beta) = (m-n)\beta f(\beta) \leq 0$ by $(f2)$ and the fact that $1 < m \leq n$. Hence $\overline{Q} \geq 0$.

Applying Lemma 2.4, noting that $Q(u(t)) \geq 0$ on $[0, \overline{R}]$, by (4.9) we have for $r \in (\overline{R}, R)$

$$\begin{aligned} mr^n E(r) &\geq P(r) \geq \int_0^r Q(u(s))s^{n-1}\,\mathrm{d}s \\ &\geq \int_0^{R_k} Q(u(s))s^{n-1}\,\mathrm{d}s + \int_{\overline{R}}^r Q(u(s))s^{n-1}\,\mathrm{d}s \\ &\geq \int_0^{R_k} Q(u(s))s^{n-1}\,\mathrm{d}s - \overline{Q}\int_{\overline{R}}^r s^{n-1}\,\mathrm{d}s \\ &\geq Q(k_2\alpha)\frac{R_k^n}{n} - \overline{Q}\frac{r^n}{n} \end{aligned}$$

$$\geq \frac{Q(k_2\alpha)}{n}\left(\frac{dn\alpha^{m-1}}{f(k_1\alpha)}\right)^{n/m} - \frac{\overline{Q}}{n}r^n,$$

where $Q(k_2\alpha) = \min_{[k\alpha,\alpha]} Q(s)$, $k_2 \in [k, 1]$.

Now assume for contradiction that $I^-$ is empty. For any number $a > 0$, define $R_a = \min\{\overline{C} + a, R\}$, where $\overline{C}$ is the number given in (3.4) We claim that $R_a \in (\overline{R}, R]$. Indeed, if $R_a = R$ there is nothing to prove; otherwise, if $R_a = \overline{C} + a < R$, then $R_a > \overline{R} + a$, since by the assumption of contradiction $\overline{R} < \overline{C}$. The claim is therefore proved.

By (3.1) and the fact that $R_a \leq \overline{C} + a$, we can fix $\alpha \geq \lambda/k$, so large that

$$E(r) \geq F(\lambda) + H(\lambda/a) \quad \text{in } (\overline{R}, R_a]. \tag{3.15}$$

It follows that $R_a \leq \overline{C} + a < R$ for this $\alpha$. Indeed, otherwise $R_a = R < \infty$ and $u'(R_a) = 0$, by Lemma 2.2-($ii$) and since $I^- = \emptyset$. Hence

$$E(R_a) = F(u(R_a)). \tag{3.16}$$

From the fact that $F(u) < F(\lambda)$ when $u < \lambda$, we get

$$F(u(r)) < F(\lambda) \quad \text{for } r \in (\overline{R}, R_a]. \tag{3.17}$$

Taking $r = R_a$ and using (3.16) gives $E(R_a) < F(\lambda)$, which contradicts (3.15). Hence $R_a \leq \overline{C} + a < R$.

Finally, (2.3), (3.15) and (3.17) imply that $H(u'(r)) > H(\lambda/a)$ and, since $H$ is strictly increasing by ($\Omega$2), it follows that $|u'(r)| = -u'(r) > \lambda/a$ on $(\overline{R}, R_a]$. This leads to

$$u(R_a) < \frac{\lambda}{a}(\overline{R} - R_a + a) = \frac{\lambda}{a}(\overline{R} - \overline{C}) < 0,$$

which contradicts the fact that $u(r) > 0$ on $[0, R)$.

Hence $I^-$ is nonempty, and in fact contains the value $\alpha > \max\{\lambda/k, 1\}$ for which (3.15) hold, completing the proof of the claim. ∎

**Lemma 3.6** *$I^-$ is open.*

**Proof.** Let $\alpha \in I^-$ and $(\alpha_j)_j$ be a sequence approaching $\alpha$ as $j \to \infty$. Let $u$ be the solution of (2.1), or (2.8), corresponding to $u(0) = \alpha$ and $u_j$ the solution corresponding to $u_j(0) = \alpha_j$, with respective maximal domains

$J = (0, R)$ and $J_j = (0, R_j)$, $R$, $R_j \le \infty$. Let $E(r)$ and $E_j(r)$ be the energy functions associated to $u$ and $u_j$, respectively.

Set $d = E(R)/2$. Clearly $d > 0$, since $\alpha \in I^-$. By Lemma 2.3 and (2.4) we can choose $r_0 \in (R/2, R)$ such that $2d < E(r_0) < 3d$ and

$$R_j > r_0, \quad d \le E_j(r_0) \le 4d, \quad u_j(r_0) \le 2u(r_0) \le \beta \tag{3.18}$$

for $j$ sufficiently large. Integrating (2.4) over $[r_0, R_j]$ yields

$$\begin{aligned} |E_j(R_j) - E_j(r_0)| &= \left| \int_{r_0}^{R_j} \frac{n-1}{r} \rho_j(r) \Omega(\rho_j(r))\,\mathrm{d}r \right| \\ &\le \frac{n-1}{r_0} \sup_{[r_0,R_j)} \Omega(\rho_j) \left| \int_{u_j(r_0)}^{u_j(R_j)} \mathrm{d}u \right| \\ &\le \frac{n-1}{r_0} u_j(r_0) \sup_{[r_0,R_j)} \Omega(\rho_j) \\ &\le \frac{4(n-1)}{R} u(r_0) \sup_{[r_0,R_j)} \Omega(\rho_j), \end{aligned} \tag{3.19}$$

with unified notation for both cases $R_j = \infty$, or $R_j < \infty$.

Moreover, since $E_j$ is decreasing, for any $r \in [r_0, R_j)$

$$H(\rho_j(r)) = E_j(r) - F(u_j(r)) \le E_j(r_0) - F(u_j(r)) \le 4d + \overline{F},$$

where $\overline{F} = \min_{[0,\beta]} |F|$, as usual. Therefore

$$\Omega(\rho_j(r)) \le \Omega(H^{-1}(4d + \overline{F})) = \overline{d},$$

and by (3.19) we obtain

$$|E_j(R_j) - E_j(r_0)| \le \frac{4(n-1)}{R} \overline{d} u(r_0).$$

This formula remains valid if we replace $r_0$ by any $r$ in $(r_0, R)$. Since $u(r) \to 0$ as $r \to R^-$, then $E_j(R_j) \ge d > 0$ by (3.18) provided $j$ is sufficiently large. Hence $R_j < \infty$, $u_j'(R_j) < 0$ and $u_j(R_j) = 0$ by Lemma 2.2-(*ii*), in other words $\alpha_j$ belongs to $I^-$ for $j$ sufficiently large, namely $I^-$ is open. ∎

**Proof of the Theorem 3.1.** In view of Lemmas 3.2-3.6 there is $\alpha > \beta$ such that

$$\alpha \notin I^+ \cup I^-.$$

We denote by $u_\alpha$ the corresponding solution of (2.1), with maximal domain $J_\alpha = (0, R_\alpha)$. Since $\alpha \notin I^+$, then $\ell_\alpha = 0$, and since $\alpha \notin I^-$, then either $R_\alpha = \infty$ or $R_\alpha < \infty$ and in both cases $u'_\alpha(R_\alpha) = 0$ by Lemma 2.2-(*ii*).

In the first case $u_\alpha$ is a classical positive radial ground state of (1.1), in the latter $u_\alpha$ is a classical radial solution of (1.2), with $R = R_\alpha$. This completes the proof. ■

**Remarks.** Under the assumptions of Theorem 3.1, if $f$ is regular with $f(0) = 0$, or if $f$ is singular with $\limsup_{u\to 0^+} f(u) < \infty$, then any solution $u$ of (1.2) of class $C^1$, extended by the value 0 for all $x$, with $|x| > R$, and still denoted by $u$, is a $C^1(\mathbf{R}^n)$ distributional compactly support ground state of (1.1).

Indeed, let $\phi \in C_0^\infty(\mathbf{R}^n)$ and let $B(0, r)$ be the ball in $\mathbf{R}^n$, centered at $x = 0$ with radius $r < R$. Multiplying (1.2) by $\phi$, and integrating by parts, we get

$$\int_{B(0,r)} A(|\nabla u|)\nabla u \cdot \nabla\phi = \int_{B(0,r)} f(u)\phi + \int_{\partial B(0,r)} A(|\nabla u|)\frac{\partial u}{\partial n}\phi.$$

Let $r \to R^-$, then the left hand side approaches a limit while the second term on the right approaches 0. Hence the first term on the right side also approaches a limit. Writing $f(u) = [f(u) - M] + M$, where $M = \sup_{(0,\alpha]} f(u) < \infty$, since $\limsup_{u\to 0^+} f(u) < \infty$, it follows easily that $f(u(r)) - M$, a nonpositive function, is in $L^1(0, R)$. Therefore $f(u(r)) \in L^1(0, R)$ and we get

$$\int_{B(0,r)} A(|\nabla u|)\nabla u \cdot \nabla\phi = \int_{B(0,r)} f(u)\phi.$$

But then, since $u(r) \equiv 0$ outside $B(0, R)$ and $f(0) = 0$, we find finally that

$$\int_{\mathbf{R}^n} A(|\nabla u|)\nabla u \cdot \nabla\phi = \int_{\mathbf{R}^n} f(u)\phi,$$

as required.

Moreover, whenever Theorem 2 of [15] can be applied, then there are no positive ground states of (1.1), radial or not. For instance, this is the case when

$$f(u) \le 0 \quad \text{for } 0 < u \le \delta, \quad \text{and} \quad \int_0^\delta \frac{\mathrm{d}s}{H^{-1}(F(s))} < \infty.$$

For a more detailed discussion see [15], [9] and also [16].

Hence, in the setting discussed above, Theorem 3.1 shows that in these special cases problem (1.1) admits a $C^1(\mathbf{R}^n)$ distributional compactly support radial ground state. More specifically, this occurs when $\limsup_{u\to 0^+} f(u) < 0$

and so, in the regular case, when $f(0) < 0$, under the assumptions of Theorem 2 of [15] and Theorem 3.1. This somewhat answers the conjecture given in Section 6.1 of [13].

The study of uniqueness of radial ground states of (1.1) or (1.2) is very delicate. The first results are contained in [6], where the function $A$ is treated in a very general setting, but the nonlinearity $f$ is assumed to have a *sublinear* growth at infinity. In [13] and [14] a uniqueness result is obtained for problems (1.1) and (1.2), for nonlinearities $f$ of type $(f1)$-$(f2)$, considered here, under the main assumption that

$$\frac{t\Omega(t)}{G(t)} \quad \text{is nondecreasing on } (0, \infty). \tag{3.20}$$

By the elliptic structure $(\Omega 2)$ of the equation

$$m = \inf_{t>0} \frac{t\Omega(t)}{G(t)} \geq 1,$$

so that in [13] and [14] also the new case $m = 1$ was covered, see the explicit example of [13].

As shown in (2.20), assumptions $(\Omega 1)$-$(\Omega 3)$ imply only that

$$\frac{t\Omega(t)}{G(t)} \geq m, \quad \text{but } m > 1.$$

For example $\Omega(t) = t^{m-1}[m(1+t/(1+t)) + t/(1+t)^2]$ verifies $(\Omega 1)$-$(\Omega 3)$, see [5], but not (3.20). Moreover (3.20) implies (2.6), namely $(\Omega 3)$. Naturally the case $m = 1$ cannot be covered in this paper.

Recently in [17], uniqueness of the solution of (1.1) and (1.2) are proved when $\Omega(t) = t^{m-1}$, $m > 1$, namely for the $m$-Laplacian case, under natural general assumptions on $f$, which overlap those of [13].

To clarify the discussion above we present the main result given in Theorem 3.1 for a canonical nonlinearity useful in many applications to physics, see for example see [11], [10] and [4], and treated for the uniqueness problem in [13] and [17].

**Corollary 3.7** *Let*

$$f(u) = -u^p + u^q, \quad -1 < p < q. \tag{3.21}$$

$(i)$ *There exists a radial ground state $u$ of (1.1) when either*

$$1 < m < n \quad \text{and} \quad 0 < p < q < m^* - 1,$$

*where*

$$\frac{1}{m^*} = \frac{1}{m} - \frac{1}{n},$$

*or*

$$m = n \quad \text{and} \quad 0 < p < q.$$

*Moreover, $u$ is positive in the entire $\mathbf{R}^n$ if and only if $p \geq m - 1$.*

($ii$) *There exists a positive radial solution of (1.2) for some $R > 0$ if either*

$$1 < m < n \quad \text{and} \quad -1 < p < m - 1, \quad q < m^* - 1,$$

*or*

$$m = n \quad \text{and} \quad -1 < p < m - 1.$$

($iii$) *If $1 < m < n$ and $q \geq m^* - 1$, then problem (1.1) admits no positive ground states, radial or not.*

**Proof.** Clearly $f$ in (3.21) satisfies ($f1$), ($f2$) and ($H2$). Moreover ($H1$) obviously holds if $m = n$, while in the main case $1 < m < n$ the principal condition ($H1$) is valid if $q < m^* - 1$.

The corollary then follows by application of Theorems 1, 2 of [15], Theorem 3.1 above and Theorem 3.2 of [12]. ■

**Remarks.** Note that condition ($H1$) fails when $f$ has exponential growth at infinity and the existence will be proved in Section 5. These nonlinearities, has been treated for the uniqueness problem in [14] and studied for the $m$-Laplacian equation in [7], [9] and in [20], [21].

We recall that for this general case any ground state of (1.1), having a unique critical point, is radial if $f$ is also regular, with $f(0) = 0$, and $f$ is nonincreasing on some interval $[0, \delta)$, $\delta > 0$, and $A$ is also assumed to be of class $C^{1,1}(0, \infty)$. We remind for a complete discussion to Theorems 2 and 3, of [18].

In particular, we note that for $A$ also of class $C^{1,1}(0, \infty)$, then every ground state $u$ of (1.1) in case ($i$) of Corollary 3.7, having a unique critical point at $x = 0$, is radially symmetric with respect to 0 and $u(r)$, $r = |x|$, has the property that $u'(r) < 0$ for all $r > 0$ such that $u(r) > 0$.

# 4 Another existence theorem

Adding an extra condition on the main divergence part of equation (1.1), we establish other criteria for existence, with proofs based on the lemmas given in Section 3. Assume that

($\Omega 4$) *the number* $\kappa = \inf_{t>0} \dfrac{\Omega(t)}{t^{m-1}} > 0$.

Of course $\Omega(t) = t^{m-1}(e^t - 1)$ verifies ($\Omega 1$)-($\Omega 3$) but not ($\Omega 4$). In this section we require that $(f1)$, $(f2)$ and ($\Omega 1$)-($\Omega 4$) hold, and still denote by $\gamma$ the number introduced in Section 3.

**Theorem 4.1** *Suppose that*

$(C1)$ $\gamma < \infty$ *and there exists* $k_0 > 0$ *such that*

$$\limsup_{u\to\gamma^-} \frac{f(u)}{(\gamma - u)^{m-1}} < k_0; \tag{4.1}$$

$(C2)$ $\gamma = \infty$ *and* $n < m$.

$(C3)$ $\gamma = \infty$ *and* $n \geq m$. *Moreover the function* $Q$ *given in Lemma 2.4 satisfies condition* $(H1)$ *of Theorem 3.1.*

*Then either (1.1) has a positive radial ground state, or (1.2) has a radial solution for some* $R > 0$. *Moreover, if* $r = |x|$, *the funtion* $u = u(r)$ *obeys* $u'(r) < 0$ *for all* $r > 0$ *such that* $u(r) > 0$.

We follow the proof of Theorem 3.1 word by word. As before we denote by $u_\alpha$ the solution of (2.8) with maximal domain $J_\alpha = (0, R_\alpha)$, $R_\alpha \leq \infty$, but now we restrict the parameter $\alpha$ to be in the interval $[\beta, \gamma)$. Also put $\ell_\alpha = \lim_{r\to R_\alpha^-} u_\alpha(r)$, and again $\ell_\alpha$ belongs to $[0, \beta)$ by Lemma 2.2-$(iii)$.

Define now

$$I^- = \{\alpha \in [\beta, \gamma) : R_\alpha < \infty,\ \ell_\alpha = 0,\ u'_\alpha(R_\alpha) < 0\},$$
$$I^+ = \{\alpha \in [\beta, \gamma) : \ell_\alpha > 0\}.$$

It is evident that $I^+$ and $I^-$ are disjoint, and that the proofs of Lemmas 3.2-3.4 and 3.6 can be carried over in all the cases $(C1)$-$(C3)$ word by word. The only change regards Lemma 3.5, whose proof now relies on assumption ($\Omega 4$).

**Lemma 4.2** $I^-$ *is nonempty.*

**Proof.** We shall prove this lemma under the three different conditions $(C1)$, $(C2)$ and $(C3)$.

*Assume* $(C1)$. Take $\varepsilon \in (0, \gamma - \beta)$ such that if $\gamma - \varepsilon \leq u < \gamma$, then

$$\frac{f(u)}{(\gamma - u)^{m-1}} < k_0 + 1,$$

and let $\lambda = \gamma - \varepsilon$. For any $\alpha \in (\lambda, \gamma)$ let $u_\alpha$ be the corresponding solution of (2.8) and let $\overline{R} = \overline{R}(\alpha)$ be the unique value of $r$ where $u(r) = \lambda$. Then

$$\lambda \le u(r) < \gamma, \quad u'(r) < 0, \quad f(u(r)) > 0 \quad \text{in } (0, \overline{R}].$$

By these inequalities, $(\Omega 4)$ and (2.8) it follows for $r \in [0, \overline{R}]$

$$\begin{aligned} \kappa \rho^{m-1}(r) &\le \Omega(\rho(r)) = \frac{1}{r^{n-1}} \int_0^r s^{n-1} f(u(s))\,\mathrm{d}s \\ &\le \frac{r}{n} \sup_{[0,r]} f(u(s)) \le \frac{r}{n}(k_0+1)[\gamma - u(r)]^{m-1}. \end{aligned}$$

In turn, for all $r \in [0, \overline{R}]$,

$$\rho(r) \le c r^{1/(m-1)}[\gamma - u(r)] \le c \overline{R}^{1/(m-1)} \left( \gamma - \alpha + \int_0^r \rho(s)\,\mathrm{d}s \right),$$

where $c = [(k_0+1)/n\kappa]^{1/(m-1)}$. Applying Gronwall's inequality, we obtain

$$\rho(r) \le c\overline{R}^{1/(m-1)}(\gamma - \alpha) e^{c\overline{R}^{m/(m-1)}}, \quad r \in [0, \overline{R}], \tag{4.2}$$

which shows that $\overline{R}(\alpha) \to \infty$ as $\alpha \to \gamma^-$. Indeed, if $\overline{R}(\alpha)$ remains bounded, then the previous inequality implies that $\max_{[0,\overline{R}]} \rho(r) \to 0$, contradicting the fact that

$$\alpha - \lambda = u(0) - u(\overline{R}) \le \overline{R} \max_{[0,\overline{R}]} \rho(r).$$

Thus (3.4) holds if $\alpha$ is sufficiently close to $\gamma$, and so $I^- \ne \emptyset$, as desired.

*Assume* $(C2)$. Fix $\lambda > \beta$ and put $\overline{C} = \overline{C}(\lambda)$ the constant defined in (3.3). Suppose for contradiction that $I^-$ is empty. Then by Lemma 3.4, for any $\alpha > \lambda$ there holds

$$\overline{R} = \overline{R}(\alpha) < \overline{C}, \tag{4.3}$$

where we recall that $\overline{C}$ is independent of $\alpha$. Define

$$v^{m-1}(r) = r^{n-1}\Omega(\rho(r)), \quad r \in J, \tag{4.4}$$

then by equation (2.8) one has $(v^{m-1})' = r^{n-1} f(u)$, and therefore $v$ is increasing on $[0, \overline{R}]$. Let $V = v(\overline{R})$, obviously $v(r) \le V$, or equivalently,

$$\rho(r) \le V^* r^{-\frac{n-1}{m-1}} \quad \text{on } [0, \overline{R}],$$

where $V^* = V\kappa^{1/(1-m)}$. Integrating this inequality over $[0, \overline{R}]$ leads to

$$\alpha - \lambda \le \frac{m-1}{m-n} V^* \overline{R}^{\frac{m-n}{m-1}}.$$

Combining this with (4.3) there results

$$V^* \ge (\alpha - \lambda) \frac{m-n}{m-1} \overline{C}^{\frac{n-m}{m-1}}. \tag{4.5}$$

Now we introduce the function

$$D(r) = \frac{m-1}{m} v^m(r) + \kappa^{1/(m-1)} r^{\tilde{m}} F(u(r)), \tag{4.6}$$

where $\tilde{m} = m(n-1)/(m-1)$. By the definition of $v$, $(\Omega 4)$ and (2.8) we have

$$\left(\frac{m-1}{m} v^m\right)' = v(v^{m-1})' \le \kappa^{1/(m-1)} r^{\tilde{m}} \rho(r) f(u),$$

and so we obtain

$$D'(r) \ge \tilde{m} \kappa^{1/(m-1)} r^{\tilde{m}-1} F(u(r)). \tag{4.7}$$

Let $r \in (\overline{R}, R)$ and integrate (4.7) on $[\overline{R}, r]$ to obtain

$$\begin{aligned} \frac{m-1}{m} v^m(r) \ge \frac{m-1}{m} V^m \quad &+ \quad \kappa^{1/(m-1)} \Big[\overline{R}^{\tilde{m}} F(\lambda) - r^{\tilde{m}} F(u(r)) \\ &+ \quad \tilde{m} \int_{\overline{R}}^{r} s^{\tilde{m}-1} F(u(s))\,\mathrm{d}s.\Big] \end{aligned}$$

We denote by $r_\beta$ the unique point such that $u(r_\beta) = \beta$. Therefore, by $(f2)$ and (3.2),

$$\begin{aligned} \overline{R}^{\tilde{m}} F(\lambda) - r^{\tilde{m}} F(u(r)) + \tilde{m} \int_{\overline{R}}^{r} s^{\tilde{m}-1} F(u(s))\,\mathrm{d}s \\ \ge -r^{\tilde{m}} F(u(r)) + \tilde{m} \int_{r_\beta}^{r} s^{\tilde{m}-1} F(u(s))\,\mathrm{d}s \\ \ge -r^{\tilde{m}} \left[F(\lambda) + \overline{F}\right]. \end{aligned}$$

Define for any constant $a > 0$

$$R_a = \min\{\overline{C} + a, R\}.$$

We assert that $R_a \in (\overline{R}, R]$. This is obvious if $R_a = R$; otherwise, if $R_a = \overline{C} + a < R$, then $R_a > \overline{R} + a > \overline{R}$ by the assumption of contradiction (4.3). Clearly $0 < u(r) < \lambda$ for $\overline{R} < r < R_a$ and we find with the help of (4.3) that

$$v^m(r) \geq V^m - \frac{m\kappa^{1/(m-1)}}{m-1}(\overline{C} + a)^{\tilde{m}}[F(\lambda) + \overline{F}], \quad r \in [\overline{R}, R_a].$$

It is clear that (4.5) implies

$$V \geq \kappa^{1/(m-1)}(\alpha - \lambda)\frac{m-n}{m-1}\overline{C}^{(n-m)/(m-1)},$$

so, if $\overline{R} < r < R_a$ and $\alpha$ is sufficiently large, we have

$$v^m(r) > [\overline{C} + a]^{\tilde{m}} \, [\Omega(\lambda/a)]^{m/(m-1)}\,, \tag{4.8}$$

in other words by (4.4)

$$r^{n-1}\Omega(\rho(r)) > [\overline{C} + a]^{n-1}\Omega(\lambda/a)$$

and in turn $\rho(r) > \lambda/a$, since $r < \overline{C} + a$. Now, if $R_a = R$, then $u'(R) = 0$ since $\alpha \notin I^-$, contradicting $u'(R) = \lim_{r \to R^-} u'(r) \leq -\lambda/a$. Thus $R_a = \overline{C}+a$, leading to $u'(r) < -\lambda/a$ for $\overline{R} < r < \overline{C}+a$. Integrating from $\overline{R}$ to $R_a = \overline{C}+a$ gives

$$u(R_a) < \lambda - \frac{\lambda}{a}(R_a - \overline{R}) = \frac{\lambda}{a}(\overline{R} - \overline{C}).$$

Since $\overline{R} < \overline{C}$ by (4.3), we obtain $u(R_a) < 0$, contradicting the fact that $u(r) > 0$ for $0 \leq r < R$. This shows that $I^-$ is not empty.

*Assume* $(C3)$. We proceed essentially as in the proof of Lemma 3.5. Let $\lambda > \beta$ be such that $Q(u) \geq 0$ if $u \geq \lambda$. Choose $\alpha > \lambda/k$, where $k$ is the constant specified in $(H1)$, and let $u_\alpha$ be the corresponding solution of (2.8), with domain $J = (0, R)$. Set $R_k$ to be the unique point where $u$ reaches $k\alpha$: clearly $R_k < \overline{R}$. Also define $\overline{\alpha} = \overline{\alpha}(k, \alpha)$ by $f(\overline{\alpha}) = \max_{[k\alpha,\alpha]} f(s)$ as in the proof of Lemma 3.5. In fact, we can take $\overline{\alpha} = k_1\alpha$ for some $k_1 \in [k, 1]$. For any $r \in (0, R_k)$, integration of (2.8) over $[0, r]$ gives by $(\Omega 4)$

$$\kappa r^{n-1}\rho^{m-1}(r) \leq r^{n-1}\Omega(\rho(r)) = \int_0^r s^{n-1} f(u(s))\,\mathrm{d}s \leq \frac{f(\overline{\alpha})}{n} r^n,$$

thus

$$\rho(r) \leq \left[\frac{f(\overline{\alpha})}{\kappa n}\right]^{1/(m-1)} r^{1/(m-1)}.$$

Integrating this over $[0, R_k]$ leads to

$$\alpha(1-k) \leq \frac{m-1}{m}\left[\frac{f(\overline{\alpha})}{\kappa n}\right]^{1/(m-1)} R_k^{1/(m-1)},$$

and therefore

$$R_k \geq \left[\frac{dn\alpha^{m-1}}{f(\overline{\alpha})}\right]^{1/m}, \quad \text{where } d = \kappa\left[\frac{(1-k)m}{m-1}\right]^{m-1}. \tag{4.9}$$

Now we can proceed, word by word, as the proof of Lemma 3.5, and the result follows. ■

The proof of Theorem 4.1 can now be completed exactly as that of Theorem 3.1.

# 5 Further results

Throughout the section we assume $(f1)$, $(f2)$ and $(\Omega 1)$-$(\Omega 3)$. The ground state $u$ we have obtained in Section 3 and 4 has the property that

$$\beta < u(0) < \gamma.$$

Since $u' < 0$ it follows that $\gamma$ is an upper bound for the $L^\infty$ norm of the ground state in the simple case $(C1)$ of Theorem 4.1. When $\gamma = \infty$ we can obtain an a priori $L^\infty$ estimate of $u$ in terms of $n$, $m$ and the nonlinearity $f$ by the proof of Lemma 3.5 or Lemma 4.2.

To treat the case $(C2)$ of Theorem 4.1, let $\lambda > \beta$ and $\overline{C} = \overline{C}(\lambda)$ be given in (3.3). Suppose $\alpha > \lambda$ is not in $I^-$, then (4.3) and (4.5) hold and we can proceed as in the proof of Lemma 4.2, subcase $(C2)$, to obtain (4.8), provided that

$$\kappa^{m/(m-1)}\left(\frac{m-n}{m-1}\right)^m \overline{C}^{m(n-m)/(m-1)}(\alpha-\lambda)^m$$
$$\geq [\overline{C}+a]^{m(n-1)/(m-1)}\left(\frac{m\kappa^{1/(m-1)}}{m-1}[\overline{F}+F(\lambda)] + \Omega(\lambda/a)^{m/(m-1)}\right),$$

or equivalently

$$\begin{aligned} \alpha > \alpha^* &= \lambda + \frac{m-1}{m-n}\overline{C}^{(m-n)/(m-1)}\left(\frac{m}{\kappa(m-1)}[\overline{F}+F(\lambda)]\right. \\ &\qquad \left. + \kappa^{m/(1-m)}\Omega(\lambda/a)^{m/(m-1)}\right)^{1/m}[\overline{C}+a]^{(n-1)/(m-1)}, \end{aligned} \tag{5.1}$$

where $a$ is any positive number. By the final argument in the proof of Case $(C2)$ of Lemma 4.2, one then reaches a contradiction. Consequently, when (5.1) is valid, then $\alpha \in I^-$ and in turn a radial ground state $u$ of (1.1), or a radial solution of (1.2), has the upper bound $\alpha^*$.

Next we consider case $(C3)$. Let $\lambda$ be such that $Q(v) \geq 0$ for $v \geq \lambda$, and $k \in (0, 1]$. Suppose $\alpha > 0$ is not in $I^-$. Then we can proceed as in the proof of Lemma 3.5 or Lemma 4.2 to obtain (3.15), provided that

$$Q(k_2\alpha)\left(\frac{dn\alpha^{m-1}}{f(k_1\alpha)}\right)^{n/m} \geq \left[\overline{Q} + nmF(\lambda) + nmH(\lambda/a)\right](\overline{C} + a)^n, \qquad (5.2)$$

with $\alpha > \max\{\lambda/k, 1\}$ or $\alpha > \lambda/k$, $k_1$,$k_2 \in [k, 1]$ and $d$ as in (3.14) or (4.9), and again $\overline{C} = \overline{C}(\lambda)$ is given in (3.3) and $a$ is any positive number. Then by the argument of Lemmas 3.5 and 4.2, case $(C3)$, one obtains a contradiction. So $\alpha \in I^-$ when (5.2) holds, and in turn the corresponding solution of (2.8) cannot be a radial ground state, or a radial solution of (1.2). This proves the following theorem.

**Theorem 5.1** *Let $u$ be a radial ground state of (1.1) or a radial solution of (1.2).*

*(i) Assume $(C2)$ and $(\Omega 4)$. Then*

$$u(r) < \alpha^* \quad \text{for all } r \geq 0,$$

*where $\alpha^*$ is the constant given in (5.1) and $\lambda$ is any number larger than $\beta$.*

*(ii) Assume that $\gamma = \infty$, $m \leq n$, $(H1)$ and $(H2)$ or $(C3)$ and $(\Omega 4)$. If there exists $\alpha_* > 0$ such that (5.2) holds for all $\alpha \geq \alpha_*$, then*

$$u(r) < \alpha_* \quad \text{for all } r \geq 0.$$

Following [9], now we consider some special cases of interest. For semplicity of notation in the next examples we denote $\operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $m > 1$, by $\Delta_m u$. Consider the equation

$$-\Delta_4 u - \Delta_8 u = -u + u^3 \quad \text{in } \mathbf{R}^3. \qquad (5.3)$$

Then $n = 3$, $m = 4$ and $\kappa = 1$ in $(\Omega 3)$, $(\Omega 4)$,

$$F(u) = -\frac{1}{2}u^2 + \frac{1}{4}u^4,$$

$\beta = \sqrt{2}$ and $\overline{F} = 1/4$. For $\lambda = 2$

$$F(2) = 2 \quad \text{and} \quad \overline{C}(2) = 6\left(\frac{\sqrt{13}+1}{6}\right)^{1/4}.$$

Moreover

$$\alpha^* = 2 + 3\overline{C}^{1/3}\left[3 + \left(\frac{2}{a}\right)^4\left[1 + \left(\frac{2}{a}\right)^4\right]^{4/3}\right]^{1/4}(\overline{C}+a)^{2/3}, \qquad (5.4)$$

taking $a = 2.58$ which approximately minimizes the right hand side of (5.4), we get $a^* \approx 31.739$. Thus any radial ground state solution $u$ of equation (5.3) satisfies

$$|u(r)| < 31.74 \quad \text{for all } r \geq 0.$$

Moreover note that for this example Corollay 3.7 shows that any ground state necessarily has compact support.

For case $(ii)$ it is more difficult to find explicit a priori estimates. Even for simple situations, the estimate (5.2) can give an extremely large value for the supremum. Consider the example

$$-\Delta_2 u - \Delta_4 u = -u + u^3 \quad \text{in } \mathbf{R}^3. \qquad (5.5)$$

Obviously $n = 3$, $m = 2$, $\kappa = 1$ and $F(u)$, $\beta$ and $\overline{F}$ are as above. We have also

$$Q(u) = -2u^2 + \frac{1}{2}u^4, \quad \overline{Q} = 2, \quad d = 2(1-k).$$

For $\lambda = 2$ then $F(2) = 2$ and $\overline{C}(2) = 3(\sqrt{19}+1)^{1/2}$. Now, taking $k_2 = k$ and $k_1 = 1$, inequality (5.2) becomes

$$3\sqrt{6}(1-k)^{3/2}k^4\alpha\frac{1-4/k^2\alpha^2}{(1-1/\alpha^2)^{3/2}} \geq \left[14 + 12\left(\frac{1}{a^2} + \frac{6}{a^4}\right)\right]\left(\overline{C}+a\right)^3. \qquad (5.6)$$

Finally choosing $a = 2.34$, which approximately minimizies the right hand of (5.6) and taking $k = 8/11$, condition (5.6) is satisfied for all $\alpha \geq 50,826$. So an a priori estimate for the supremum of the ground state for (5.5) is

$$|u(r)| < 50,826 \quad \text{for all } r \geq 0.$$

The case

$$-\Delta_2 u - \Delta_4 u = -u + u^3 \quad \text{in } \mathbf{R}^2 \qquad (5.7)$$

is also instructive. Here $n = m = 2$ and $\beta$, $\overline{F}$, $F$, $d$ are the same as for (5.5), while $Q(v) = 4F(v)$ and $\overline{Q} = 1$. Again let $\lambda = 2$, so $F(2) = 2$, $\overline{C}(2) = \frac{3}{2}(\sqrt{19}+1)^{1/2}$ and (5.2) becomes

$$4(1-k)k^4\alpha^2\frac{1-2/k^2\alpha^2}{1-1/\alpha^2} \geq \left[9+\frac{8}{a^2}+\frac{32}{a^4}\right]\left(\overline{C}+a\right)^2. \tag{5.8}$$

Choosing $a = 2.16$ and $k = 4/5$, condition (5.8) is satisfied for all $\alpha > 34.35$, which is a much more resonable value. Thus an a priori estimate for the supremum of the ground state in this case is

$$|u(r)| < 34.35 \quad \text{for all } r \geq 0.$$

We conclude these numerical experiments analyzing another operator, namely $-\Delta_2 - \varepsilon\Delta_6$, proposed by Benci, Fortunato and Pisani in [2] as a model in quantuum mechanics. Consider

$$-\Delta_2 u - \varepsilon\Delta_6 u = -u + u^3 \quad \text{in } \mathbf{R}^3, \tag{5.9}$$

with $\varepsilon = 1/100$. Thus $n = 3$, $m = 2$, $\kappa = 1$, moreover $F$, $\beta$ and $\overline{F}$ are as above. We have also

$$Q(u) = -2u^2 + \frac{1}{2}u^4, \quad \overline{Q} = 2, \quad d = 2(1-k).$$

For $\lambda = 2$ then $F(2) = 2$ and $\overline{C}(2) = 9/\Phi^{-1}(9/2) = 4.55465$, where here $\Phi(t) = t^2 + \varepsilon t^6$. Now, taking $k_2 = k$ and $k_1 = 1$, inequality (5.2) becomes

$$3\sqrt{6}(1-k)^{3/2}k^4\alpha\frac{1-4/k^2\alpha^2}{(1-1/\alpha^2)^{3/2}} \geq \left[14+\frac{12}{a^2}+\frac{3.2}{a^6}\right]\left(\overline{C}+a\right)^3. \tag{5.10}$$

The value $a = 1.18$ approximately minimizies the right hand side of (5.10), so that, taking $k = 8/11$, condition (5.10) is satisfied for all $\alpha \geq 3,361$. Hence an a priori estimate for the supremum of the ground state for (5.9) is

$$|u(r)| < 3,361 \quad \text{for all } r \geq 0.$$

Theorem 4.1 shows a striking difference between the cases $m \leq n$ and $n < m$. In fact there is also an important difference between the cases $m = n$ and $m < n$. When $m < n$ we have already noted in Corollary 3.7 the importance of the subcritical requirement $q < m^* - 1$ for the nonlinearity (3.21). When

$m = n$, however, there is no critical Sobolev exponent $m^*$, and no optimal embedding $W^{1,m}(\mathbf{R}^n) \hookrightarrow L^{m^*}(\mathbf{R}^n)$. Instead the appropriate embedding is into an Orlicz space (see [19]), and critical growth means exponential growth. For the classical field equation such exponential behavior for $f$ was treated, for example, in [3] and [1]. For a complete discussion on the background and literature we remind to [7] and [9].

On the other hand Theorem 3.1 (and also Theorem 4.1) is not directly satisfactory for exponential growth, since condition (3.1) fails. Now we discuss more precisely such cases.

As in the proof of Theorem 3.1 (or in case $(C3)$ of Theorem 4.1), in order to show that a value $\alpha$ is in $I^-$ it is enough to verify (5.2). Assume hereafter that $n = m$, so $Q(v) = n^2 F(v)$, and without loss of generality we take $\lambda \geq \max\{\beta, 1\}$. Put $k_1\alpha = \overline{\alpha}$, $k_2 = k$ and $(1-k)\alpha = \rho$, so that the condition $\alpha > \lambda/k$ is equivalent to $\alpha > \lambda + \rho$. By (3.3) at once (5.2) reduces to

$$\rho^{n-1} \frac{F(\alpha - \rho)}{f(\overline{\alpha})} > \Gamma = \Gamma(\beta, \lambda, a, n), \tag{5.11}$$

with

$$\Gamma = B\left[\overline{F} + F(\lambda) + H\left(\frac{\lambda}{a}\right)\right]\left\{\frac{n\lambda[1 + \overline{F}/F(\lambda)]}{\Phi^{-1}(\frac{n}{n-1}[\overline{F} + F(\lambda)])} + a\right\}^n, \tag{5.12}$$

where $\overline{\alpha}$ is an arbitrary number in $[\alpha - \rho, \alpha]$ and either

$$B = \min\left\{\frac{n}{[\Omega^{-1}(1)]^{n-1}}, \frac{n\Omega(1-k)}{(1-k)^{n-1}}\right\}$$

if $(H1)$ and $(H2)$ hold, or

$$B = \frac{(n-1)^{n-1}}{\kappa n^n}$$

if $(C3)$ and $(\Omega 4)$ hold.

So we have proved the following result

**Theorem 5.2** *Assume $n = m$ and $(H1)$-$(H2)$ or $(C3)$ and $(\Omega 4)$. Suppose also that there exist constants $\lambda > \beta$ (also that $\lambda > \max\{\beta, 1\}$ in the first case), $\rho > 0$ and $\alpha > \lambda + \rho$ such that*

$$\rho^{n-1} \frac{F(\alpha - \rho)}{f(\overline{\alpha})} > \Gamma, \tag{5.13}$$

*where $\overline{\alpha}$ is any number in $[\alpha - \rho, \alpha]$ and $\Gamma$ is given in (5.12).*

*Then the claim of either Theorem 3.1 or Theorem 4.1 holds.*

Note that the number in (5.11) depends only on the behavior of $f$ for large $\alpha$, and $\Gamma$ depends on the behavior of $f$ on $[0, \lambda]$.

We shall now apply Theorem 5.2 when the nonlinearity $f$ has exponential growth at infinity. Suppose that $q \geq 0$, $h \geq 0$ and

$$f(s) = \omega(s)e^{hs^q} \quad \text{for } s > \tau, \tag{5.14}$$

where $\omega$ is a locally Lipschitz continuous function for $s > \tau$, with

$$\omega_1 s^{p_1} \leq \omega(s) \leq \omega_2 s^{p_2}, \tag{5.15}$$

$\omega_1 > 0$, $\omega_2 > 0$ and $p_1 \leq p_2$. Then

**Corollary 5.3** *Assume $n = m$ and either $(H1)$-$(H2)$, or $(C3)$ and $(\Omega 4)$. Suppose also that (5.14) and (5.15) are satisfied. Then the claim of either Theorem 3.1, or Theorem 4.1 holds, provided that $0 \leq q < 1 - (p_2 - p_1)/n$.*

**Proof.** Indeed, as shown in Theorem 3.1 of [9], condition (5.14) and (5.15) give

$$\lim_{\alpha \longrightarrow \infty} \rho^{n-1}(\alpha) \frac{F(\alpha - \rho(\alpha))}{f(\overline{\alpha})} = \infty,$$

where $\rho(\alpha) = \alpha^{1-q}$ if $q > 0$ and $\rho(\alpha) = \alpha/2$ if $q = 0$. Consequently condition (5.11) holds and the corollary is proved. ■

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