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PRECISE DAMPING CONDITIONS FOR GLOBAL ASYMPTOTIC STABILITY FOR NONLINEAR SECOND ORDER SYSTEMS, II

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§1. Introduction.

We shall be concerned with the vibrations of holonomic dynamical systems with N degrees of freedom, subject to variable nonlinear damping. From a physical standpoint, it is reasonable to expect that damping a differential system will cause transient effects to decay to zero as time increases. Our purpose is to present precise conditions guaranteeing that such decay in fact occurs, and to do so even for the general case of quasi–Lagrangian systems of the form given in (1.1) below.

This paper is one of a series on the asymptotic stability of nonlinear systems. In an earlier work [18] we studied the global asymptotic stability of systems whose damping is, roughly speaking, not bounded away from zero as a function of time. Here we consider the complementary situation, that is, when the damping is bounded away from zero outside of any given neighborhood $V_1 \cup V_2$ in the $2N$ dimensional phase space (u, p) of the system, where V_1, V_2 are respectively neighborhoods of the manifolds $u = 0$ and $p = 0$ in the phase space.

We allow the damping to be unbounded in magnitude. In this case it is well known that global asymptotic stability can fail due to the phenomenon of overdamping. In order to obtain asymptotic stability it is therefore necessary to place appropriate growth restrictions on the damping as the time increases to infinity. Our concern will be with establishing such restrictions in generality, and moreover in such a way that *the restrictions apply only for certain subsets I of times tending to infinity*. Thus in particular the damping is left uncontrolled for the remaining set of times. The concept of a partially controlled damping has been considered so far only for scalar equations and for subsets I of fairly special form ([22], [24], [1], [10]). In this respect our work is significantly different from previous studies.

Previous work in the scalar case, that is with only 1 degree of freedom, is due to Smith, Levin & Nohel, Artstein & Infante, Surkov, and especially Thurston & Wong, Ballieu & Peiffer and Hatvani. For systems with $N > 1$ degrees of freedom, in addition to the elementary case of Rayleigh damping we are aware only of work of Duffin and Salvadori, both of whom however consider only the case of bounded damping.

Our method of proof differs from previous ones, being based on a Liapunov–type variational identity introduced in [16]. No appeal, however, will be made to results obtained in previous papers or to known theorems on differential equations.

While our concerns are theoretical, the phenomenon of nonlinear damping is naturally also of physical interest. In addition to standard electrical and mechanical networks, which can be represented in quasi–Lagrangian form, see [7] as well as standard treatises on analytical mechanics, other examples arise in reactor dynamics [13], in the tempering and heat treatment of materials and in poisoning of catalysts [1].

We consider vector unknowns $u:\, J \to \mathbb{R}^N$, and systems having the form

$$\big(\nabla\mathcal{L}(t,u,u')\big)' - \nabla_u\mathcal{L}(t,u,u') = Q(t,u,u'), \quad t\in J, \tag{1.1}$$

where J is a half open interval of the form $[T,\infty)$ and $\mathcal{L}(t,u,p) = G(u,p) - F(t,u)$, and where

$$G:\, \mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}, \qquad F:\, J\times\mathbb{R}^N \to \mathbb{R}, \qquad Q:\, J\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^N$$

are given continuously differentiable functions. The most important of the conditions which will be imposed on (1.1) are that

$$G(u,\cdot) \;\text{ is strictly convex in } \mathbb{R}^N; \qquad G(u,0)=0, \quad \nabla G(u,0)=0, \tag{1.2}$$

$$\big(\nabla_u F(t,u),u\big) > 0 \quad \text{for } u\neq 0; \qquad F(t,0)=0, \tag{1.3}$$

$$\big(Q(t,u,p),p\big) \le 0. \tag{1.4}$$

Here $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$ and

$$\nabla = \nabla_p = \left(\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_N}\right), \qquad \nabla_u = \left(\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_N}\right).$$

The function F represents a restoring potential, this being analytically described by (1.3), while Q represents a general nonlinear damping, expressed by (1.4). In Section 2 we shall give a complete set of hypotheses. Explicit examples are given in [18] and [12].

Since

$$\nabla G(u,0) = \nabla_u G(u,0) = \nabla_u F(t,0) = Q(t,u,0) = 0$$

it is clear that the rest state $u\equiv 0$ is a solution of (1.1). This state is said to be *a global attractor* for the system if any bounded solution u, defined on some interval J, has the property

$$u(t),\; u'(t) \to 0 \quad \text{as } t\to\infty.$$

This concept of asymptotic stability is defined with respect to *bounded* solutions rather than with respect to *all* solutions. The motivation for this point of view is that the concepts of boundedness and of stability are essentially different and accordingly should be treated separately (for the boundedness of solutions see [3], [4], [17] and [24]).

The special system

$$u'' + A(t, u, u')u' + f(u) = 0, \tag{1.5}$$

is particularly important, it being (for $N = 1$) the main object of the studies [1–3], [5], [8–10], [13] and [22–24]. In the scalar case $A(t, u, p)$ is a non–negative continuous real function, while for $N > 1$ it is taken as a non–negative definite continuous matrix. In turn (1.5) can be identified with (1.1) by the natural choices $\mathcal{L}(t, u, p) = \frac{1}{2}|p|^2 - F(u)$ and $Q(t, u, p) = -A(t, u, p)p$.

Now suppose for this system that there are continuous control functions

$$\varphi : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty) \qquad \text{and} \qquad \delta : J \to (0, \infty),$$
$$\text{with} \quad \varphi(u, p) > 0 \quad \text{for } u \neq 0,\ p \neq 0,$$

and numbers γ, $q > 0$ such that

$$\varphi(u, p) \le |A(t, u, p)p| \cdot |p| \le \gamma \big(A(t, u, p)p, p\big), \qquad \big|A(t, u, p)p\big| \le \delta(t)\,|p|$$

for $t \in J$, $u \in \mathbb{R}^N$ and $|p| \le q$. Also, we suppose in accord with (1.3) that

$$\big(f(u), u\big) > 0 \quad \text{for } u \neq 0.$$

Then, as immediate consequences of our main results, the following conclusions hold for the system (1.5).

THEOREM A. *Suppose*

$$1/\delta \in BV(J), \qquad 1/\delta \notin L^1(J). \tag{1.6}$$

Also, when $N > 1$, assume for all $U > 0$ and $p_0 > 0$ that there is a measurable function $h : J \to [0, \infty)$ with $h \notin L^1(J)$, such that

$$\big(A(t, u, p)p, p\big) \ge h(t) \tag{1.7}$$

for $t \in J$, $|u| \le U$ and $|p| \ge p_0$.

Then the rest state is a global attractor for (1.5).

THEOREM B. *Suppose that*

$$1/\delta \in \mathrm{Lip}^-(J), \qquad 1/\delta \notin L^1(J). \tag{1.8}$$

Also, when $N > 1$*, assume that* (1.7) *holds and that*

$$|A(t,u,p)p - a(t,u,p)p| = o(1) \quad \textit{as } p \to 0 \tag{1.9}$$

uniformly for $t \in J$ *and* u *in compact subsets of* $\mathbb{R}^N$*, where* a *is a non–negative continuous function in* $J \times \mathbb{R}^N \times \mathbb{R}^N$*.*

Then the rest state is a global attractor for (1.5)*.*

We emphasize that for the scalar case it is only conditions (1.6) and (1.8) which are needed.

Theorem A is a consequence of Corollary 2 in Section 3, with $m = 2$, $\mu = 1$. Theorem B similarly follows from Corollary 1 in Section 3, with the same values of m and μ. Condition (1.7) is exactly (V_1) in Section 2, while (1.9) is condition (2.8) in hypothesis (V_2). The remaining hypotheses required for the application of Corollaries 1 and 2 are easy consequences of the assumptions stated above.

Theorems A and B are new in both the scalar and vector case of (1.5). When (1.5) is scalar, some results related to Theorems A and B had previously been given by Ballieu & Peiffer, see the comments at the end of Section 3. In the scalar case of (1.5), and for A depending only on t, some of our results overlap with recent work of Hatvani & Totik. Further results for (1.5) as well as for the general system (1.1) can be obtained by direct application of the main theorems in Sections 3 and 8. The case of intermittently controlled damping will be the object of a more extended study in the forthcoming paper [19].

The following section contains the principal hypotheses on G, F and Q. In Section 3 the main results of the paper are stated; their proofs are given in Sections 4–6. The final sections (7 and 8) contain several generalizations of our main theorems and various additional comments. Further discussion of previous work and its relation with ours can be found in references [18] and [19].

Note added in proof. A modified version of the main Theorems 1–3 is contained in the following paper by Leoni in this journal, with furthermore a partially simplified proof technique.

§2. Principal hypotheses.

We consider vector solutions $u = (u_1, \dots, u_N)$ of the quasi–Lagrangian ordinary differential system

$$(\nabla G(u,u'))' - \nabla_u G(u,u') + f(t,u) = Q(t,u,u'), \quad J = [T,\infty), \tag{2.1}$$

where ∇ denotes the gradient operator with respect to the variable p and

$$f(t,u) = \nabla_u F(t,u). \tag{2.2}$$

It will be supposed throughout the paper that

$$G \in C^1(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{R}), \quad F \in C^1(J \times \mathbb{R}^N;\mathbb{R}), \quad Q \in C(J \times \mathbb{R}^N \times \mathbb{R}^N;\mathbb{R}^N)$$

and also that the following natural conditions hold:

(H_1) $G(u,\cdot)$ is strictly convex in $\mathbb{R}^N$ for all $u \in \mathbb{R}^N$, with $G(u,0) = 0$ and $\nabla G(u,0) = 0$.

(H_2) $F(t,0) = 0$ for all $t \in J$. For all u_0, U with $0 < u_0 < U$ there exists a constant $\kappa > 0$ and a non–negative function $\psi \in L^1(J)$ such that

$$\big(f(t,u),u\big) \geq \kappa \qquad \text{when } t \in J \qquad \text{and} \qquad |u| \in [u_0,U], \tag{2.3}$$

$$|F_t(t,u)| \leq \psi(t) \qquad \text{when } t \in J (\text{a.e.}) \quad \text{and} \qquad |u| \leq U. \tag{2.4}$$

(H_3) $(Q(t,u,p),p) \leq 0$ for all $t \in J$ and u, $p \in \mathbb{R}^N$. There is a continuous function $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ with

$$\varphi(u,p) > 0 \quad \text{when } u \neq 0 \text{ and } p \neq 0,$$

and for all $U > 0$ there exists a constant $q = q(U) > 0$, such that

$$|(Q(t,u,p),p)| \geq \varphi(u,p) \quad \text{for } t \in J,\ |u| \leq U \text{ and } |p| \leq q. \tag{2.5}$$

When F does not depend on t, the inequality (2.3) follows from the condition $(f(u),u) > 0$ for $u \neq 0$, while (2.4) is irrelevant. If $Q = Q(u,p)$ and $(Q(u,p),p) < 0$ for $u, p \neq 0$, it is easy to see that (2.5) is satisfied.

In the scalar case condition (2.2) is automatic, e.g. we can take

$$F(t,u) = \int_0^u f(t,s)\,ds.$$

The conditions $G(u,0) = 0$, $(Q(t,u,p),p) \leq 0$ and (2.3) respectively imply $\nabla_u G(u,0) = Q(t,u,0) = f(t,0) = 0$.

The next hypothesis places in evidence the concept of a *control set* $I \subset J$ where the damping term Q is subject to a growth restriction.

(H_4) For every $U > 0$ there exist a measurable control set $I \subset J$, a positive measurable damping function $\delta : I \to (0,\infty)$ and a continuous weight function ρ, with $\rho(0) = 0$, such that

$$(Q(t,u,p),u) \leq \delta(t)\rho(p) \quad \text{for } t \in I,\ |u| \leq U,\ |p| \leq q. \tag{2.6}$$

Although φ, δ and ρ may depend on U, for simplicity we do not specifically indicate this dependence.

Hypotheses (H_1)–(H_4) are all that is needed when $N = 1$. For the vectorial case $(N > 1)$ we require some further assumptions.

(V_1) For all $U > 0$ and $p_0 > 0$ there is a non–negative measurable function $h \notin L^1(J)$ such that

$$|(Q(t,u,p),p)| \geq h(t) \quad \text{for } t \in J,\ |u| \leq U,\ |p| \geq p_0. \tag{2.7}$$

(V_2) For all $U > 0$ there exists a continuous function $\varepsilon(p)$ with $\varepsilon(0) = 0$, such that

$$(Q(t,u,p),u) \leq \varepsilon(p) \tag{2.8}$$

when $t \in J$, $|u| \leq U$, $|p| \leq q$ and $(\nabla G(u,p),u) \geq 0$. Moreover there exists a C^1 function $g(u)$ and a constant $C \geq 0$ such that

$$\frac{(\nabla_u g(u),p)}{|p|} - \frac{(\nabla G(u,p),u)}{|\nabla G(u,p)|} \leq C\,\frac{\varphi(u,p)}{|p|} \tag{2.9}$$

when $|u| \leq U$, $0 < |p| \leq q$ and $(\nabla G(u,p),u) < 0$.

(V_3) For all $U > 0$ there exists $\gamma \geq 1$ such that

$$|p| \cdot |Q(t,u,p)| \leq \gamma\, |(Q(t,u,p),p)| \qquad \text{for } t \in I,\ |u| \leq U,\ |p| \leq q. \tag{2.10}$$

Here $q = q(U)$ and φ in (V_2) and (V_3) are the functions introduced in condition (H_3).

Conditions (2.8) and (V_3) relate the unit vectors $p/|p|$, $Q/|Q|$ and $\nabla G/|\nabla G|$. It is easy to see, for example, that (V_3) holds if $p/|p|$ and $Q/|Q|$ are bounded away from orthogonality, and that (2.8) is satisfied when ∇G and $-Q$ have the same direction.

Condition (2.9) is more delicate. Although it is a needed technical assumption, nevertheless it places only a mild restriction on the form of G; thus it is satisfied in the majority of important cases, as shown by the following

LEMMA 2.1. *Condition* (2.9) *is valid when*

(i) $N = 1$;
(ii) $\nabla G(u,p)$ *has the same direction as* p;
(iii) $\nabla G(u,p)$ *has the same direction as* $\mathbf{A}p$, *where* $\mathbf{A}$ *is a positive definite symmetric matrix;*
(iv) *there is a function* $R \in C^2(\mathbb{R}^N;\mathbb{R})$ *with positive definite Hessian matrix* $R_{uu}(u)$, *such that* $\nabla G(u,p)$ *has the same direction as* $R_{uu}(u)p$;
(v) $G = \hat{G}(p)$, *where* $\hat{G}$ *satisfies* (H_1), *is of class* C^3 *for* p *near zero and* $(\partial^2\hat{G}/\partial p_i\partial p_j)$ *is positive definite at* $p = 0$; *here we require also the condition* $\varphi(u,p) \geq \mathrm{Const.}\ |p|^2$.

Remark. The canonical actions $G(p) = |p|^m/m$, $m > 1$ and $G(p) = \sqrt{1+|p|^2} - 1$ all fall under various cases of this lemma.

Proof. When $N = 1$, $\nabla G(u,p)$ necessarily has the same direction as p, so (i) is a special case of (ii). Also (ii) is the special case $\mathbf{A} = \mathbf{I}$ of (iii), and (iii) is the special case $R(u) = \frac{1}{2}(\mathbf{A}u, u)$ of (iv). Hence we must prove only (iv) and (v).

To obtain (iv), take

$$g(u) \equiv C_1 R^*(u), \tag{2.11}$$

where C_1 is a positive constant to be determined, and $R^*(u) = (\nabla_u R(u), u) - R(u)$ is the Legendre transform of R. Then

$$(\nabla_u g(u), p) = C_1 (R_{uu}(u)u, p),$$

$$\frac{(\nabla G(u,p), u)}{|\nabla G(u,p)|} = \frac{(R_{uu}(u)p, u)}{|R_{uu}(u)p|} = \frac{(R_{uu}(u)u, p)}{|R_{uu}(u)p|} = \frac{(\nabla_u g(u), p)}{C_1 |R_{uu}(u)p|}. \tag{2.12}$$

Now suppose $(\nabla G(u,p), u) < 0$. Hence also $(\nabla_u g(u), p) < 0$. Let $\lambda_1(u)$ be the least eigenvalue of the positive definite matrix $R_{uu}(u)$ and

$$\lambda_1 = \min_{|u| \le U} \lambda_1(u) > 0.$$

Therefore

$$\frac{\lambda_1 (\nabla_u g(u), p)}{|R_{uu}(u)p|} \ge \frac{(\nabla_u g(u), p)}{|p|}.$$

Then taking $C_1 = 1/\lambda_1$ in (2.11) and using (2.12), it is clear that (2.9) holds with $C = 0$.

(v) Here, by Taylor's formula, we obtain for $p \to 0$

$$\nabla G(p) = \mathbf{A}p + O(|p|^2), \quad \text{where } \mathbf{A} = \left(\frac{\partial^2 \hat{G}}{\partial p_i \partial p_j}(0) \right).$$

Then choosing

$$g(u) = \tfrac{1}{2} \|\mathbf{A}^{-1}\| (\mathbf{A}u, u)$$

(as in case (iv)) we get, for $|u| \le U$ and $p \to 0$,

$$D \equiv \frac{(\nabla_u g(u), p)}{|p|} - \frac{(\nabla G(p), u)}{|\nabla G(p)|} = (\mathbf{A}p, u) \left\{ \frac{\|\mathbf{A}^{-1}\|}{|p|} - \frac{1}{|\mathbf{A}p|} \right\} + O(|p|). \tag{2.13}$$

But when $(\nabla G(p), u) < 0$ and $|u| \le U$ it is clear that $(\mathbf{A}p, u) \le \text{Const. } |p|^2$. Therefore, since the quantity in braces on the right side of (2.13) is non-negative and $\le \text{Const.}/|p|$, we obtain

$$D \le \text{Const. } |p| \le \text{Const. } \varphi(u,p)/|p|,$$

as required.

Because of the term $\varphi(u,p)/|p|$ on the right side of (2.9), the conditions (ii)–(iv) of the lemma need be satisfied only in the limit as $p \to 0$, provided the convergence is suitably strong (as in case (v)). Summing up, one can expect (2.9) to hold for functions $G(u,p)$ having the typical structure

$$G(u,p) = \tilde{G}(u,P)\,[1+o(1)] \qquad \text{as } p \to 0,$$

where $P = P(u,p)$ has the form

$$P(u,p) = (R_{uu}(u)p,p).$$

This obviously covers a great variety of actions G, including, as we have already noted, all standard operators as well as the scalar case $N=1$.

In the next section we denote by $\mathrm{Lip}^-(J)$ the set of functions $k : J \to \mathbb{R}$ such that the lower right Dini derivate, $D_r^- k$, is bounded below on J.

Remark. The hypothesis (H_3) requires a lower bound on the damping which is *independent* of t. As we have noted in the introduction, a condition of this type is basic to the considerations of this paper, in contrast to the assumptions made in [18]. Nevertheless, (H_3) can be significantly weakened at the expense of a slightly more delicate discussion. For expositional purposes we defer this discussion until Section 7.

§3. Main results.

Our principal stability results are stated in this section. We assume throughout that conditions (H_1)–(H_4) are satisfied. In stating the theorems we assume that δk is extended to all of J by the definition $\delta(t)k(t) = 0$ for $t \in J\backslash I$.

THEOREM 1. *Let $N \geq 1$, and assume (V_1) is satisfied when $N > 1$. Suppose that there is a non–negative bounded continuous function k on J such that*

$$k \notin L^1(J), \qquad k = 0 \ \text{ on } J \setminus I, \tag{3.1}$$

$$k \in BV(J) \quad \text{or} \quad \log k \in \mathrm{Lip}(J), \tag{3.2}$$

and

$$\liminf_{t\to\infty} \int_T^t \delta(s)k(s)\exp\left(-\int_s^t k(r)\,dr\right) ds < \infty. \tag{3.3}$$

Then the rest state $u \equiv 0$ *is a global attractor for the system* (2.1).

Remark. In the case when $\log k \in \mathrm{Lip}(J)$ we must have k positive on J, hence in turn $I = J$.

An immediate special case occurs when (2.6) holds in the stronger form

$$(Q(t,u,p),u) \le \rho(p) \quad \text{for } t \in J,\ |u| \le U,\ |p| \le q,$$

that is when $I = J$ and $\delta(t) \equiv \text{Const.} > 0$ in (H_4). Then we can take $k \equiv 1$ and the hypotheses (3.1)–(3.3) are all obviously satisfied.

Theorem 1 can be improved if the functions G and Q have additional structure. In particular, we introduce the following further conditions, which sharpen (H_1) and (H_4):

(1) There exist a positive constant Θ and an exponent $m > 1$ such that

$$|\nabla G(u,p)| \le \Theta |p|^{m-1} \quad \text{when } |u| \le U \text{ and } |p| \le 1. \tag{3.4}$$

(2) Assumption (H_4) is satisfied with the specific function

$$\rho(s) = |s|^{\mu}, \quad \mu > 0. \tag{3.5}$$

Obviously (3.4) holds when $G(u,p) = |p|^m/m$, with $\Theta = 1$. It is also satisfied when $G(u,p) = \sqrt{1+|p|^2} - 1$, with $\Theta = 1$ and $m = 2$. Another example, in which G depends explicitly on u, is $G(u,p) = (1+|u|^a)|p|^m/m$ where $a > 0$, $m > 1$.

THEOREM 2. *Let* (3.4) *and* (3.5) *hold, and assume, when* $N > 1$, *that* (V_1)–(V_3) *are satisfied. Suppose also that there is a non–negative bounded continuous function* k *on* J *such that* (3.1) *and*

$$\begin{aligned} k^{m-1} &\in \mathrm{Lip}^-(J) \qquad \textit{if} \quad 1 < m \le 2 \\ k &\in \mathrm{Lip}^-(J) \qquad \textit{if} \quad m > 2 \end{aligned} \tag{3.6}$$

are satisfied.

Finally assume

$$\liminf_{t\to\infty} \int_T^t \delta(s) k^{\mu+1}(s) \exp\left(-\int_s^t k(r)\,dr \right) ds < \infty. \tag{3.7}$$

Then the rest state is a global attractor for the system (2.1).

Theorem 2 improves Theorem 1 not only by replacing the restrictive hypothesis (3.2) by the better assumption (3.6), but also by allowing an extra factor k^{μ} in the integral (3.3).

THEOREM 3. *When $m > 2$ let* (2.5) *be strengthened to*

$$|(Q(t,u,p),p)| \geq \sigma(t)\tau(u)|p|^{\nu+1} \quad \text{for } t \in J,\ |u| \leq U \ \text{and}\ |p| \leq q, \tag{3.8}$$

where $\nu > 0$, $\sigma(t) \geq 1$ and τ is a continuous function on $\mathbb{R}^N$ with $\tau(u) > 0$ for $u \neq 0$.

Then Theorem 2 remains true when $(3.6)_2$ *is replaced by the weaker condition*

$$D_r^- k \geq -\text{Pos. Const.} \begin{cases} \sigma^{(m-2)/\nu}, & \nu > m-2 \\ \sigma, & 0 < \nu \leq m-2. \end{cases} \tag{3.9}$$

Because of the change from (2.5) to (3.8) in Theorem 3, it is necessary to define the corresponding function φ appearing in (2.9). We make the natural choice $\varphi(u,p) = \tau(u)|p|^{\nu+1}$.

COROLLARY 1. *Suppose* (3.4) *and* (3.5) *are satisfied. Also let* (H_4) *hold with $I = J$ and*

$$1/\delta \notin L^{1/\mu}(J),$$

where δ is continuous and

$$\begin{aligned} \delta^{(1-m)/\mu} &\in \mathrm{Lip}^-(J) \qquad \text{if} \quad 1 < m \leq 2, \\ \delta^{-1/\mu} &\in \mathrm{Lip}^-(J) \qquad \text{if} \qquad m > 2. \end{aligned} \tag{3.10}$$

Assume finally, when $N > 1$, that (V_1)–(V_3) *are verified.*

Then the rest state is a global attractor for (2.1).

Proof. Choose $k = \delta^{-1/\mu}$. Then (3.1), (3.6) are satisfied, and k is bounded (see Lemma 4.3 (ii)). Moreover, $\delta k^\mu = 1$. Hence the integral in (3.7) is equal to

$$\int_T^t k(s) \exp\left(-\int_s^t k(r)\,dr \right) ds.$$

But this can be explicitly integrated, with the value

$$1 - \exp\left(-\int_T^t k(r)\,dr \right) \leq 1.$$

Consequently (3.7) is satisfied, and Theorem 2 can be applied.

COROLLARY 2. *The result of Corollary 1 continues to hold even when the hypotheses* (3.4) *and* (V_2) *are dropped, provided* (3.10) *is replaced by* $\delta^{-1/\mu} \in CBV(J)$ *or* $\log \delta \in \mathrm{Lip}\,(J)$.

This follows from Theorem 4 in Section 8, again with $k = \delta^{-1/\mu}$.

For the special case $N = 1$, $G(p) = p^2/2$, $\varphi(u,p) = \tau(u)|p|$ and $\mu = 1$, Balleiu & Peiffer obtained results closely related to these. In their work, in place of the condition $\tau(u) > 0$ for $u \neq 0$ which is required here, they use the slightly weaker relations

$$\tau(u) \geq 0, \qquad \int_{-u}^{u} \tau(s)\,ds > 0 \quad \text{for } u > 0. \tag{3.11}$$

On the other hand, their condition (b), [2, p. 325], namely

$$-\theta \leq \delta'(t)/\delta^2(t) \leq \text{Const.}, \quad \theta \in (0,1),$$

is stronger than (3.10), which for the present parameter values $m = 2$, $\mu = 1$ takes the form

$$\delta'(t)/\delta^2(t) \leq \text{Const.}$$

In the scalar case, by combining the ideas of Ballieu & Peiffer [2, p. 322] with those of the present paper, we can allow the function φ in (H_3) to have the alternative form

$$\varphi(u,p) = \tau(u)\phi(p),$$

where $\tau(u)$ obeys (3.11) and $\phi(p) > 0$ for *all* $p \neq 0$, without affecting our conclusions.

Finally, we have the following consequence of Theorem 3, when $k = \delta^{-1/\mu}$.

COROLLARY 3. *Let* (3.8) *hold when* $m > 2$. *Then Corollary 1 remains true when* $(3.10)_2$ *is replaced by*

$$D_r^+\delta \leq \text{Const.} \begin{cases} \delta^{(\mu+1)/\mu}\sigma^{(m-2)/\nu} & \nu > m-2 \\ \delta^{(\mu+1)/\mu}\sigma & 0 < \nu \leq m-2. \end{cases} \tag{3.12}$$

In the "natural" case $\mu = \nu \geq m-2$, $\sigma = \delta$, condition (3.12) reduces to

$$\delta^{(1-m)/\mu} \in \mathrm{Lip}^-(J),$$

which is the same as the first relation in (3.10). An example here is the variational problem

$$\delta \int_J r(t) \left\{ \frac{1}{m}|u'|^m - F(t,u) \right\} dt = 0,$$

where $m > 1$ and $r \in C^1(J;(0,\infty))$. The Euler–Lagrange system for this problem can be written in the quasi–Lagrangian form (2.1) with

$$G(u,p) = \frac{1}{m}|p|^m, \quad Q(t,u,p) = -a(t)|p|^{m-2}p, \quad a(t) = \frac{r'(t)}{r(t)}.$$

Our hypotheses are then satisfied when $a(t) \geq \text{Const.} = a_0 > 0$, with δ any continuous function such that

$$\frac{a(t)}{a_0} \leq \delta(t) \leq \theta\,\frac{a(t)}{a_0}, \quad \theta \geq 1,$$

and with

$$\sigma(t) = \delta(t), \quad \rho(s) = a_0 U|s|^{m-1}, \quad \tau(u) = a_0/\theta, \quad \mu = \nu = m-1, \quad \Theta = 1.$$

When $N > 1$ also (V_1)–(V_3) are satisfied with

$$h(t) = a_0 p_0^m, \qquad \varepsilon(p) = 0, \qquad g(u) = \tfrac{1}{2}|u|^2, \qquad C = 0, \qquad \gamma = 1.$$

Then $(3.10)_1$ and (3.12) hold if $1/\delta \in \mathrm{Lip}^-(J)$, and Corollary 3 applies provided $1/\delta \notin L^{1/(m-1)}(J)$.

Further consequences of our theorems, especially for the case when I is a proper subset of J, will be given in the forthcoming paper [19].

§4. Preliminary lemmas, I.

Before giving the main proofs in Section 6 we present here some elementary consequences of the assumptions in Section 2.

LEMMA 4.1. *Suppose* (H_1) *holds and let* $H(u,\cdot)$ *be the Legendre transform of* $G(u,\cdot)$, *namely*

$$H(u,p) = (\nabla G(u,p),p) - G(u,p). \tag{4.1}$$

Then

$$\inf\{H(u,p) : |u| \leq U,\ |p| \geq p_0\} = h_0(U,p_0) > 0 \tag{4.2}$$

for all $U,\ p_0 > 0$.

Proof. Fix u, p in $\mathbb{R}^N$ with $p \neq 0$ and define $g(t) = G(u,tv)$, $t \geq 0$, where $v = p/|p|$. Obviously g is strictly convex and of class C^1 with $g(0) = 0$. Then for $0 \leq t_0 < t$ we have

$$\begin{aligned}[t g'(t) - g(t)] - [t_0 g'(t_0) - g(t_0)] &= g'(\xi)(t_0 - t) + t g'(t) - t_0 g'(t_0)\\ &= t\{g'(t) - g'(\xi)\} + t_0\{g'(\xi) - g'(t_0)\},\end{aligned}$$

where $\xi \in (t_0, t)$. But g' is strictly increasing, so that the last expression is strictly positive. Taking $t_0 = 0$ gives $tg'(t) - g(t) > 0$, so that $H(u,p) > 0$ for $p \neq 0$. Moreover, if $|p| \geq p_0 > 0$, then

$$H(u,p) = |p|g'(|p|) - g(|p|) \geq p_0 g'(p_0) - g(p_0) = H(u, p_0 v).$$

Hence (4.2) holds with

$$h_0(U, p_0) = \min\{H(u,p) \ : \ |u| \leq U, \ |p| = p_0\} > 0.$$

LEMMA 4.2. *Assume that* (H_2) *holds. Then*

(i) $\inf\{F(t,u) : t \in J, \ |u| \in [u_0, U]\} > 0$ *for* $0 < u_0 \leq U$.

(ii) *Let* $u \in \mathbb{R}^N$ *be such that*

$$F(t,u) \geq C \quad \textit{for all } t \geq T_1,$$

where $C > 0$ *and* $T_1 \geq T$. *Then there exists* $u_0 > 0$, *depending only on* C *and* T_1, *such that* $|u| \geq u_0$.

Proof. (i) Since $F(t,0) = 0$, we have for $t \in J$ and $|u| \in [u_0, U]$

$$\begin{aligned} F(t,u) &= \int_0^1 \frac{d}{ds} F(t,su)\, ds = \int_0^1 (\nabla_u F(t,su), u)\, ds \\ &\geq \int_{1/2}^1 \frac{1}{s}(f(t,su), su)\, ds \geq \tfrac{1}{2}\kappa\left(\tfrac{1}{2}|u_0|, U\right) \qquad \text{by (2.3).} \end{aligned}$$

(ii) Here, for $t \geq T_2$,

$$F(t,u) - F(T_2, u) = \int_{T_2}^t F_t(s,u)\, ds \leq \int_{T_2}^\infty \psi(s)\, ds$$

by (2.4). Choose $T_2 \geq T_1$ such that $\int_{T_2}^\infty \psi(s)\, ds \leq C/2$. Consequently

$$F(T_2, u) \geq C/2.$$

Since $F(T_2, 0) = 0$ it follows that $|u| \geq u_0$ for some $u_0 = u_0(C, T_2) > 0$. The required conclusion now follows since T_2 depends only on C and T_1.

LEMMA 4.3. (i) *The function* $\varphi(u,p)$ *in* (H_3) *is* $o(|p|)$ *as* $p \to 0$, *uniformly for* $|u| \le U$.

(ii) *The function* δ *in* (H_4) *is bounded from zero on* I, *namely* $\delta(t) \ge d > 0$.

Proof. From (2.5) for $t_0 \in J$ fixed, we obtain for all $|u| \le U$ and $|p| \le q$

$$0 \le \varphi(u,p) \le |Q(t_0,u,p)| \cdot |p|.$$

The first assertion now follows from the fact that $Q(t_0,\cdot,\cdot)$ is uniformly continuous on compact sets of $\mathbb{R}^{2N}$ and $Q(t_0,u,0)=0$.

For the second assertion, take $p \in \mathbb{R}^N$ such that $|p| = \min\{q,U\} = r$. Then setting $u=-p$ in (2.5) and (2.6) we get for all $t \in I$

$$0 < \varphi(-p,p) \le |(Q(t,-p,p),p)| = (Q(t,u,p),u) \le \delta(t)\,\rho(p).$$

Hence $\delta(t) \ge d = \min\{\,\varphi(-p,p)/\rho(p) \,:\, |p| = r\} > 0$, as required.

LEMMA 4.4. *Let* k *be a non–negative continuous function in* $\mathrm{Lip}^-(J)$. *Then for every constant* $\theta > 1$ *there exists a function* $h \in C^1(J)$ *and an open set* $E \subset J$ *such that*

(i) $\theta k \ge h \ge \begin{cases} k & \textit{in } J\backslash E \\ 0 & \textit{in } E; \end{cases}$

(ii) $\int_E k \le 1$,

(iii) $\inf\limits_J h'(t) \ge \min\Big\{0,\ \theta \inf\limits_J D_r^- k(t)\Big\}$.

Proof. This follows exactly as in the proof of Lemma A given in the Appendix of [18], except that the relation (A1) in that proof must be appropriately modified. For this we may assume without loss of generality that $\inf\limits_J h' < 0$, for otherwise (iii) obviously holds. Then we get

$$\text{(A1)}' \qquad \inf_J h' = \inf_J D_r^- \,\overset{=}{k} \ge \inf_J D_r^-(\theta k),$$

as required.

Remark. This technical lemma is used only for Theorems 2, 3 and 5. Moreover it can be omitted even in these cases provided that k is assumed from the outset to be $AC(J)$ rather than $C(J)$.

§5. Preliminary lemmas, II.

Conditions (H_1)–(H_3) will be used as standing assumptions for the results of this section, without further notice.

In what follows, we consider (weak) solutions of (2.1) on J, namely vector functions $u : J \to \mathbb{R}^N$ of class C^1 for which

$$\nabla G(u(t), u'(t)) \in C^1(J; \mathbb{R}^N),$$

and which satisfy the system (2.1) in J. Let $H(u, \cdot)$ be the Legendre transform of $G(u, \cdot)$. Then for any (weak) solution of (2.1) on J we have

$$\{H(u, u') + F(t, u)\}' = (Q(t, u, u'), u') + F_t(t, u); \tag{5.1}$$

this is easily seen if $u \in C^2(J)$, but holds equally for weak solutions under the conditions of Section 2, see [20].

In what follows we denote by u any bounded solution of (2.1) and by $U(> 0)$ a fixed upper bound for the set $\{|u(t)| : t \in J\}$.

LEMMA 5.1. (i) *There exists* $\ell \geq 0$ *such that*

$$H(u, u') + F(t, u) \to \ell \quad \textit{as} \quad t \to \infty,$$

(ii) $(Q(t, u, u'), u') \in L^1(J)$,

(iii) $\varphi(u, u') \in L^1(J')$, *where*

$$J' = \{t \in J \ : \ |u'(t)| \leq q\}$$

and $q = q(U) > 0$ *is the constant given in* (H_3).

Proof. From (5.1) we get

$$H(u, u') + F(t, u) = \int_T^t \{(Q(s, u, u'), u') + F_t(s, u)\}\, ds + \text{Const.}$$

Since H, $F \geq 0$ by Lemmas 4.1 and 4.2, and since also $(Q, p) \leq 0$ and $|F_t| \leq \psi$, it follows by letting $t \to \infty$ that (ii) holds. But then in turn we get (i). Finally, from (2.5) and another use of (ii) we obtain (iii).

LEMMA 5.2. *If* $\ell = 0$ *in Lemma* 5.1(i), *then* $u(t)$, $u'(t) \to 0$ *as* $t \to \infty$.

Proof. In this case both $H(u, u') \to 0$ and $F(t, u) \to 0$ as $t \to \infty$. The required conclusion now follows from Lemmas 4.1 and 4.2.

LEMMA 5.3. *If $N = 1$, or if $N > 1$ and* (V_1) *holds, then $u'(t) \to 0$ as $t \to \infty$.*

Proof. By Lemma 5.2 it is enough to consider the case $\ell > 0$. We claim first that

$$\liminf_{t\to\infty} |u'(t)| = 0. \tag{5.2}$$

If $N = 1$ this is obvious since u is bounded. If $N > 1$ and (5.2) fails, then there is a number $p_0 > 0$ such that $|u'(t)| \ge p_0$ for all t sufficiently large. Hence by (V_1) there exists $h \notin L^1(J)$ for which

$$|(Q(t, u, u'), u')| \ge h(t)$$

for all t sufficiently large. This contradicts Lemma 5.1(ii), proving the claim.

Now assume for contradiction that $u'(t)$ does not tend to zero as $t \to \infty$. Let η be a small positive number satisfying the following three conditions:

$$\eta < q, \qquad \eta < \limsup_{t\to\infty} |u'(t)|$$

and

$$H(u, p) \le \tfrac{1}{2}\ell \quad \text{for all } |u| \le U \text{ and } |p| \le \eta,$$

where q is the number given in (H_3) corresponding to U. This is certainly possible, since $q > 0$, $\ell > 0$ and $H(u, 0) = 0$. Let $\theta \in (0, 1)$; it is clear that $|u'(t)|$ passes endlessly over the interval $[\theta\eta, \eta]$ as $t \to \infty$.

On any such individual passage, taking place in the time interval $[t_1, t_2]$, we have by Lemma 5.1(i)

$$F(t, u) \ge \frac{3}{4}\ell - H(u, u') \ge \frac{\ell}{4}$$

provided that t_1 is sufficiently large. Consequently by Lemma 4.2(ii)

$$|u(t)| \ge u_0 > 0 \quad \text{for } t \in [t_1, t_2],$$

where t_1 could be even larger, if necessary. In turn

$$\int_{t_1}^{t_2} \varphi(u, u')\, dt \ge \frac{\varphi_0}{\eta} \int_{t_1}^{t_2} |u'|\, dt,$$

where $\varphi_0 = \min\{\varphi(u, p) : |u| \in [u_0, U],\ |p| \in [\theta\eta, \eta]\} > 0$.

By Lemma 5.1(iii) the integrals $\int_{t_1}^{t_2} |u'| dt$ on successive passages must tend to zero; recall that $|u'(t)| \le \eta < q$ on $[t_1, t_2]$ so that the union of all the intervals $[t_1, t_2]$ is contained in J'. Hence

$$|u(t_2) - u(t_1)| \le \int_{t_1}^{t_2} |u'|\, dt \to 0 \tag{5.3}$$

on successive passages.

On the other hand, for $t_2 > t_1 \geq T_1 \geq T$,

$$F(t_2, u(t_2)) - F(t_1, u(t_1)) = F(T_1, u(t_2)) - F(T_1, u(t_1)) + \int_{T_1}^{t_2} F_t(s, u(t_2))\, ds - \int_{T_1}^{t_1} F_t(s, u(t_1))\, ds.$$

By choosing T_1 appropriately large, it then follows from (2.4) and (5.3) that

$$F(t_2, u(t_2)) - F(t_1, u(t_1)) \to 0$$

on successive passages. Consequently by Lemma 5.1(i)

$$H(u(t_2), u'(t_2)) - H(u(t_1), u'(t_1)) \to 0$$

on successive passages.

Now let

$$h_1 = \max\{H(u,p)\ :\ |u| \leq U,\ |p| = \theta\eta\}, \qquad h_2 = \min\{H(u,p)\ :\ |u| \leq U,\ |p| = \eta\}.$$

Clearly h_1, $h_2 > 0$. Also $\theta \in (0,1)$ can be chosen so small that $h_1 \leq h_2/2$. Hence

$$|H(u(t_2),\, u'(t_2)) - H(u(t_1),\, u'(t_1))| \geq h_2 - h_1 \geq h_1 > 0,$$

which is the required contradiction.

LEMMA 5.4. *Let $N = 1$, or if $N > 1$ let* (V_1) *hold. Suppose $\ell > 0$ in Lemma 5.1(i). Then there exist constants u_0, $\kappa > 0$ such that*

$$|u(t)| \geq u_0 \quad \textit{and} \quad (f(t, u(t)),\, u(t)) \geq \kappa \tag{5.4}$$

for all t sufficiently large.

Proof. By Lemmas 5.1(i) and 5.3 we have $F(t, u(t)) \to \ell > 0$ as $t \to \infty$. This implies (5.4), in view of Lemma 4.2(ii) and (2.3).

LEMMA 5.5. *Suppose $N = 1$, or when $N > 1$ that* (V_1)–(V_2) *hold. If $\ell > 0$ then*

$$(\nabla G(u(t),\, u'(t)),\, u(t)) < 0 \tag{5.5}$$

for all t sufficiently large. Moreover

$$[g(u)]' \in L^1(J), \tag{5.6}$$

where g is the function given in (V_2) *when $N > 1$, and $g(u) = \frac{1}{2}u^2$ when $N = 1$.*

Proof. Define $w(t) = (\nabla G(u(t), u'(t)),\, u(t))$ for $t \in J$. Then from (2.1)

$$w'(t) = (\nabla G(u, u'), u') + (\nabla_u G(u, u'), u) + (Q(t, u, u'), u) - (f(t, u), u). \tag{5.7}$$

If t is sufficiently large and $w(t) \geq 0$, we assert that $w'(t) \leq \text{Const.} < 0$.

To see this, first observe by Lemma 5.3 and (H_1) that

$$(\nabla G(u, u'), u') \to 0 \quad \text{and} \quad (\nabla_u G(u, u'), u) \to 0 \tag{5.8}$$

as $t \to \infty$. When $N = 1$ and $w(t) \geq 0$ we get $(Q(t, u(t), u'(t)), u(t)) \leq 0$, by (H_1) and (H_3). When $N > 1$ and $w(t) \geq 0$, then from (2.8) there follows

$$(Q(t, u(t), u'(t)), u(t)) \leq \varepsilon(u'(t)) \leq \kappa/2 \tag{5.9}$$

for t sufficiently large, where κ is the constant given in (5.4). The assertion now follows from (5.7)–(5.9) and (5.4).

This in turn implies that $w(t) < 0$ for all sufficiently large t, in other words (5.5) holds. Indeed, if $w(\tau) \leq 0$ for some τ sufficiently large then obviously $w(t) < 0$ for all $t > \tau$. On the other hand, if $w(\tau) > 0$ then by the assertion we should have $w(\tau_1) = 0$ at some $\tau_1 > \tau$. But then $w(t) < 0$ for $t > \tau_1$.

Even more, since $\nabla G(u, 0) = 0$, we have $u'(t) \neq 0$ whenever (5.5) holds. Now, when $N > 1$, we obtain from (2.9) and (5.5) that

$$0 > w(t) = (\nabla G(u, u'), u) \geq \frac{|\nabla G(u, u')|}{|u'(t)|}\{(\nabla_u g(u), u') - C\varphi(u, u')\} \tag{5.10}$$

provided t is sufficiently large, say $t \geq T_2$. Consequently

$$C\varphi(u, u') - [g(u)]' > 0 \quad \text{for } t \geq T_2. \tag{5.11}$$

On the other hand,

$$\begin{aligned} g(u(T_2)) - g(u(t)) &= -\int_{T_2}^{t} [g(u)]'\, ds \\ &= \int_{T_2}^{t} \{C\varphi(u, u') - [g(u)]'\}\, ds - C\int_{T_2}^{t} \varphi(u, u')\, ds. \end{aligned} \tag{5.12}$$

The left hand side of (5.12) is bounded since u is bounded. Hence from Lemma 5.1(iii) and (5.11) we see that $C\varphi(u,u') - [g(u)]' \in L^1[T_2,\infty)$. Using Lemma 5.1(iii) once more yields $[g(u)]' \in L^1(J)$, as required.

When $N = 1$ condition (2.9) holds with $g(u) = \frac{1}{2}u^2$ and $C = 0$ (see the lemma in Section 2), so the preceding argument continues to apply. This completes the proof.

The idea in the proof of looking at successive passages of $|u'(t)|$ over an interval is drawn from the work of Ballieu & Peiffer (cf. [2], Theorem 1, page 322), though they treated only (1.5) in the scalar case $N = 1$, and also considered successive passages of the function $|u(t)|$ rather than $|u'(t)|$.

§6. Proofs of the main theorems.

We first prove Theorem 1. Let u be a bounded solution of (2.1), with $|u(t)| \le U$ for $t \in J$. By Lemma 5.3 we already know that $u'(t) \to 0$ as $t \to \infty$. If $\ell = 0$ then also $u(t) \to 0$ by Lemma 5.2, and we are done.

Thus assume that $\ell > 0$. We shall show that this cannot occur, which will complete the proof. First consider the case when $k \in AC(J)$ and define

$$\omega(t) = \exp \int_T^t k(s)\, ds, \quad t \in J.$$

In view of (5.1) and (2.1), the following identity holds a.e. along $u = u(t)$ and for all $\alpha \in \mathbb{R}$, see also [15],

$$\begin{aligned}
&\{\omega[H(u,u') + F(t,u) + \alpha k(\nabla G(u,u'),u)]\}' \\
&\qquad = \omega F_t(t,u) + \omega'[H(u,u') + F(t,u) - \alpha(f(t,u),u)] \\
&\qquad\quad + \alpha\omega'[(\nabla G(u,u'),u') + (\nabla_u G(u,u'),u)] \\
&\qquad\quad + \omega(Q(t,u,u'),u') + \alpha\omega'(Q(t,u,u'),u) \\
&\qquad\quad + \alpha\omega''(\nabla G(u,u'),u) = R(t).
\end{aligned} \tag{6.1}$$

For the rest of the proof of Theorem 1 (though not for the following proofs) we shall take $\alpha = 1$.

We now estimate the various terms in $R(t)$, using throughout that

$$\omega' = k\omega, \qquad \omega'' = (k^2 + k')\,\omega. \tag{6.2}$$

1. $|F_t(t,u(t))| \le \psi(t)$ by (2.4).

2. $H(u,u') + F(t,u) = \ell + o(1)$ as $t \to \infty$ by Lemma 5.1(i).

3. $(f(t,u),u) \ge \kappa > 0$ for t sufficiently large, by (5.4).

4. By Lemma 5.3, and since $\nabla_u G(u,0) = 0$,

$$(\nabla G(u,u'),u') + (\nabla_u G(u,u'),u) = o(1) \quad \text{as } t \to \infty.$$

5. $(Q(t,u,u'),u') \le 0$ by (H_3).

6. By (2.6) and Lemma 5.3, for all $t \in I$ sufficiently large,

$$\omega'(Q(t,u,u'),u) \le k\,\omega\,\delta\,\rho(u'). \tag{6.3}$$

In fact this holds also for *all* $t \in J$ sufficiently large, since on the set $J \setminus I$ we have $\omega' = k\omega = 0$ by $(3.1)_2$ and $\delta k = 0$ by agreement. Recalling that ρ is continuous and $\rho(0) = 0$, we can therefore write (6.3) in the form

$$\omega'(Q(t,u,u'),u) \le \varepsilon_1\,\delta\,k\,\omega,$$

where $\varepsilon_1 = \varepsilon_1(t) \to 0$ as $t \to \infty$.

7. Using (6.2), we obtain for all t sufficiently large

$$\begin{aligned}\omega''(\nabla G(u,u'),u) &= (k^2 + k')\,\omega\,(\nabla G(u,u'),u)\\ &\le (k + |k'|)\omega\,\varepsilon_2 = (|k'|\omega + \omega')\,\varepsilon_2,\end{aligned}$$

where $\varepsilon_2 = \varepsilon_2(t) \to 0$ as $t \to \infty$, since $\nabla G(u,0) = 0$.

Combining the estimates of steps *1–7* now yields

$$R(t) \le \omega\{\psi + \varepsilon_1\delta k + \varepsilon_2|k'|\} + \omega'\{\ell - \kappa + \varepsilon_3\} \tag{6.4}$$

where $\varepsilon_1(t)$, $\varepsilon_2(t)$, $\varepsilon_3(t)$ are $o(1)$ as $t \to \infty$.

The rest of the proof is essentially the same as that of Theorem 3.1 in [18]. For the convenience of the reader we indicate the main steps.

First, by (3.3) there exists a positive constant M and a sequence (t_i) with $t_i \nearrow \infty$ such that, for all i,

$$\frac{1}{\omega(t_i)}\int_T^{t_i} \delta k\omega\,ds \le M. \tag{6.5}$$

Now consider the case when $k \in BV(J)$. We can choose $R \ge T$ so that

$$\varepsilon_1(t) \le \kappa/5M, \quad \varepsilon_2(t) \le 1, \quad \varepsilon_3(t) \le \kappa/5 \qquad \text{for } t \ge R; \tag{6.6}$$

$$\int_R^\infty \psi ds \le \tfrac{1}{5}\kappa, \qquad \int_R^\infty |k'|ds \le \tfrac{1}{5}\kappa. \tag{6.7}$$

Also recall our initial assumption that $k \in AC(J)$. Then from (6.1), (6.4) and (6.6), it follows that the absolutely continuous function

$$\Psi(t) = \omega(t)\Big\{H(u,u') + F(t,u) + k\,(\nabla G(u,u'),u) - (\ell - \kappa + \kappa/5) - \frac{1}{\omega(t)}\int_R^t \psi\omega\,ds - \frac{1}{\omega(t)}\int_R^t |k'|\omega\,ds - \frac{\kappa}{5M\omega(t)}\int_T^t \delta k\omega\,ds\Big\}$$

is decreasing in $[R,\infty)$. On the other hand, by Lemma 5.3, (H_1), and the boundedness of u and k,

$$k\,(\nabla G(u,u'),u) \to 0 \quad \text{as } t \to \infty.$$

Therefore from (6.5), (6.7) and Lemma 5.1(i)

$$\Psi(t_i) \geq \omega(t_i)\left\{\tfrac{1}{5}\kappa + o(1)\right\} \quad \text{as } i \to \infty,$$

which is a contradiction since $\omega(t) \to \infty$ by $(3.1)_1$.

Next, for the case when $\log k \in \mathrm{Lip}(J)$ there holds

$$|k'|\omega \leq \text{Const. } \omega\, k = \text{Const. } \omega' \quad \text{a.e. in } J.$$

Thus (6.4) reduces to

$$R(t) \leq \omega\{\psi + \varepsilon_1 \delta k\} + \omega'\{\ell - \kappa + \varepsilon_4\}.$$

The rest of the proof is essentially the same as before, and in fact even simpler since in the definition of Ψ the second integral is dropped and so $(6.7)_2$ is no longer required.

When $(3.2)_1$ holds, but k is not $AC(J)$, we proceed as in the final part of the proof of Theorem 3.1 in [18]. When $(3.2)_2$ is satisfied then $k \in \mathrm{Lip}(J)$ since k is bounded, and so k is automatically in $AC(J)$.

The proof of Theorem 2 differs from that of Theorem 1 only in Steps *6* and *7*, in the fact that $\alpha \in (0,1)$ will be chosen later, and in the use of the approximation Lemma 4.4 when k is not $AC(J)$. The new estimates *6* and *7* are as follows:

6. Here by (6.2)

$$\alpha\omega'\,(Q(t,u,p),u) = \alpha\, k\,(Q(t,u,p),u)\,\omega.$$

Now from Young's inequality, for $p \neq 0$, $\mu > 0$,

$$\alpha k \leq \frac{\mu}{\mu+1}|p| + \frac{1}{\mu+1}\frac{(\alpha k)^{\mu+1}}{|p|^{\mu}}.$$

Hence for $t \in J$, $|u| \le U$ and $|p| \le q$

$$\begin{aligned} \alpha\, k\,(Q(t,u,p),u) &\le |p|\cdot|(Q(t,u,p),u)| + (\alpha k)^{\mu+1}\delta && \text{by } (\mathrm{H}_4) \text{ and } (3.5)\\ &\le \gamma\, U\,|(Q(t,u,p),p)| + (\alpha k)^{\mu+1}\delta && \text{by } (\mathrm{V}_3); \end{aligned}$$

here we have again used $k = \delta k = 0$ on $J \setminus I$. Consequently, by Lemma 5.3,

$$\alpha\omega'\,(Q(t,u,u'),u) \le \omega\{\gamma\, U\,|(Q(t,u,u'),u')| + \alpha^{\mu+1}\delta\, k^{\mu+1}\}$$

for all t sufficiently large.

7. As in the previous estimate for this step, but now also using (5.5), (5.10) and (5.11),

$$\begin{aligned} \omega''(\nabla G(u,u'),u) &\le -(k')^{-}\omega(\nabla G(u,u'),u)\\ &\le (k')^{-}\omega\frac{|\nabla G(u,u')|}{|u'|}\{C\varphi(u,u') - [g(u)]'\} \qquad (6.8)\\ &\le \Theta(k')^{-}\omega|u'|^{m-2}\{C\varphi(u,u') - [g(u)]'\} \end{aligned}$$

by (3.4), provided t is sufficiently large. (The same result obviously holds for $N = 1$, with $C = 0$ and $g(u) = \frac{1}{2}u^2$.)

If $m > 2$ then (3.6) and (6.8) yield

$$\omega''(\nabla G(u,u'),u) \le \text{Const. } \omega\{\varphi(u,u') + |g(u)'|\}.$$

On the other hand, if $1 < m \le 2$ then again by (3.6) and (6.8)

$$\omega''(\nabla G(u,u'),u) \le \text{Const. } \left(k/|u'|\right)^{2-m}\omega\{\varphi(u,u') + |g(u)'|\}.$$

Subcase 1. $|u'| > \alpha k$. Then

$$\omega''(\nabla G(u,u'),u) \le \text{Const. } \alpha^{m-2}\omega\{\varphi(u,u') + |g(u)'|\}.$$

Subcase 2. $|u'| \le \alpha k$. Here from the first inequality of (6.8) together with (3.4) and (3.6)

$$\begin{aligned} \omega''(\nabla G(u,u'),u) \le (k')^{-}\omega\,|(\nabla G(u,u'),u)| &\le \text{Const. } k^{2-m}\omega\,|u'|^{m-1}\\ &\le \text{Const. } k^{2-m}\omega\,\alpha^{m-1}k^{m-1} = \text{Const. } \alpha^{m-1}\omega'. \end{aligned}$$

Combining the previous estimates *1–5* with the new results *6–7* gives

$$R(t) \le \omega\{\psi + \alpha^{\mu+1}\delta k^{\mu+1} + \text{Const. } \chi\} + \omega'\{\ell - \alpha\kappa + \text{Const. } \alpha^m + o(1)\} \qquad (6.9)$$

as $t \to \infty$, where

$$\chi = \chi(t) = |(Q(t,u,u'),u')| + |g(u)'| + \varphi(u,u') \in L^1(J) \tag{6.10}$$

by Lemmas 5.1 and 5.5. The rest of the proof is essentially the same as before, except that $\alpha \in (0,1)$ is first chosen appropriately small and then t sufficiently large.

When k is not $AC(J)$ we use the approximation Lemma 4.4 and proceed again as in the final part of the proof of Theorem 3.1 in [18].

Remark. If k in Theorems 1 and 2 is assumed to be absolutely continuous, rather than simply being continuous, then obviously the final part of the proof is not required.

The proof of Theorem 3 is different from that of Theorem 2 only in the case $m > 2$ of Step *7*.

8. Case $m > 2$. We suppose throughout the discussion that $|u'(t)| \le \min\{1, q\}$ for all values t under consideration.

Subcase 1. $|u'|^{\nu-m+2} \ge (k')^-/\sigma$, $\nu > m-2$. Then by (3.4) and (5.5)

$$k'(\nabla G(u,u'),u) \le \Theta U\,\sigma \frac{(k')^-}{\sigma}|u'|^{m-1} \le \Theta U\,\sigma |u'|^{\nu+1}.$$

On the other hand, by (3.8) and (5.4),

$$|(Q(t,u,u'),u')| \ge \tau_0 \sigma |u'|^{\nu+1}$$

where $\tau_0 = \min\{\tau(u) : |u| \in [u_0, U]\} > 0$. Choose

$$\alpha \le \tau_0/\Theta U,$$

so that

$$\alpha k'(\nabla G(u,u'),u) + (Q(t,u,u'),u') \le 0. \tag{6.11}$$

This estimate replaces Step *5*.

Subcase 2. $|u'|^{\nu-m+2} \le (k')^-/\sigma$, $\nu > m-2$. As in (6.8)

$$k'\,(\nabla G(u,u'),u) \le \Theta (k')^- |u'|^{m-2}\{C\varphi(u,u') + |g(u)'|\},$$

where $\varphi(u,p) = \tau(u)|p|^\nu$, see the comment following Theorem 3 in Section 3. Moreover

$$(k')^-|u'|^{m-2} \le \sigma[(k')^-/\sigma]^{1+(m-2)/(\nu-m+2)} \le \text{Const.}$$

in view of $(3.9)_1$. Hence

$$k'(\nabla G(u,u'),u) \le \text{Const. } \{\varphi(u,u') + |g(u)'|\}.$$

Combining these two subcases with Steps *1–4* in the proof of Theorem 1, and Steps *6* and *7* in Theorem 2, gives (6.9) and (6.10) as before, when $m > 2$ and $\nu > m-2$.

When $0 < \nu \le m-2$ we have $(k')^-/\sigma \le$ Const. by $(3.9)_2$, and the argument of *Subcase 1* immediately gives (6.11).

The proof is now completed exactly as in the case of Theorem 2.

§7. A weaker hypothesis.

In condition (2.5) the function $|(Q(t,u,p),p)|$ is asserted to have a lower bound $\varphi(u,p)$ which is independent of t. This requirement can be significantly weakened, as shown first by Hatvani [9] in the case $N=1$.

In particular, one can replace $\varphi(u,p)$ in (2.5) by the function

$$\sigma(t)\varphi(u,p)$$

where $\sigma(t)$ is non–negative and measurable on J, and satisfies the *positive mean value criterion*

$$\int_L \sigma(t)\,dt \ge a(|L|) > 0 \qquad \text{for all intervals } L\subset J \text{ such that } 0<|L|<1, \tag{7.1}$$

where $a(\cdot)$ is a positive function defined on $(0,1)$. A non–trivial example where (7.1) is satisfied can be constructed by choosing a sequence $t_i \nearrow \infty$ with $t_{i+1}-t_i \ge 2d > 0$, and an increasing function $b(t)$ such that $b(0)=0$ and $b(t)>0$ for $t>0$. We can then take

$$\sigma(t) \ge \min\{b(d),\ b(|t-\bar{t}|)\}$$

where $\bar{t}$ is the nearest t_i to the value t.

In general, as this example shows, condition (7.1) allows σ to have the behavior

$$\liminf_{t\to\infty}\ \sigma(t) = 0.$$

This, in turn, requires in condition (2.9) that the right hand side should be zero.

Before showing that our conclusions remain unchanged under the weakened condition (2.5), it is convenient to prove the simple

LEMMA. *For each* $\lambda\in(0,1)$

$$\frac{1}{|L|}\int_L \sigma(t)\,dt \ge \frac{1}{2\lambda}a(\lambda) \qquad \textit{when } L\subset J \textit{ and } |L|\ge\lambda. \tag{7.2}$$

Proof. Suppose first that $\lambda \le |L| < 2\lambda$. Then by (7.1), for any interval $L'\subset L$ with $|L'|=\lambda$ we get

$$\frac{1}{|L|}\int_L \sigma(t)\,dt \ge \frac{1}{|L|}\int_{L'} \sigma(t)\,dt \ge \frac{1}{|L|}a(|L'|) > \frac{1}{2\lambda}a(\lambda), \tag{7.3}$$

as required.

If $2\lambda \le |L| < 4\lambda$ then

$$\frac{1}{|L|}\int_L \sigma(t)\,dt = \frac{1}{|L|}\left(\int_{L'} \sigma(t)\,dt + \int_{L''} \sigma(t)\,dt\right)$$

where $L = L' \cup L''$ is a partition of L, with $|L'| = |L''| = \frac{1}{2}|L|$. Clearly $|L'|,\, |L''| \in [\lambda,\, 2\lambda)$. Thus by (7.3)

$$\frac{1}{|L'|}\int_{L'} \sigma(t)\,dt \ge \frac{1}{2\lambda}a(\lambda), \qquad \frac{1}{|L''|}\int_{L''} \sigma(t)\,dt \ge \frac{1}{2\lambda}a(\lambda)$$

so that obviously $\frac{1}{|L|}\int_L \sigma(t)\,dt \ge a(\lambda)/2\lambda$. Continuing this procedure for $4\lambda \le |L| < 8\lambda$, etc. completes the proof.

We now show that the weakened condition (H_3), that is, with the function $\varphi(u,p)$ replaced by $\sigma(t)\,\varphi(u,p)$, is sufficient for the purposes of the paper. Specifically, (H_3) was previously applied only in the proofs of:

Lemma 4.3(i), which remains valid as one can easily see from its proof by taking t_0 so that $\sigma(t_0) > 0$;

Lemma 5.1(iii), whose assertion is now replaced by $\sigma\,\varphi(u,u') \in L^1(J')$;

The principal Lemma 5.3, where the integrals $\int_{t_1}^{t_2} |u'|\,dt$ were shown to approach zero on successive passages. We use at this point a slightly different approach, as follows:

Suppose for contradiction that the integrals $\int_{t_1}^{t_2} |u'|\,dt$ do *not* approach zero on successive passages. Then there would be an infinite subsequence of these intervals – which we henceforth consider exclusively – such that

$$\int_{t_1}^{t_2} |u'|\,dt \ge c \qquad \text{for some positive constant } c. \tag{7.4}$$

In turn, since $\theta\eta \le |u'| \le \eta$ on each passage, we get $t_2 - t_1 \ge c/\eta$. Then, assuming without loss of generality that $c < \eta$,

$$\int_{t_2}^{t_1} |u'|\,dt \le \eta(t_2 - t_1) \le \frac{2c}{a(c/\eta)}\int_{t_1}^{t_2} \sigma(t)\,dt \tag{7.5}$$

by (7.2) with $\lambda = c/\eta$. Next, with φ_0 defined as in the original proof, there follows

$$\int_{t_1}^{t_2} \sigma(t)\,dt \le \frac{1}{\varphi_0}\int_{t_1}^{t_2} \sigma(t)\varphi(u,u')\,dt \;\to\; 0$$

on successive passages, since the union of all the intervals $[t_1,\, t_2]$ is contained in J' and $\sigma\,\varphi(u,u') \in L^1(J')$, as noted above. Hence by (7.5)

$$\int_{t_1}^{t_2} |u'|\,dt \;\to\; 0$$

on successive passages, contradicting (7.4).

Remark. For the case $N = 1$, condition (7.1) can in some circumstances be improved by using well known disconjugacy results for second order ordinary differential equations (see [9]).

It almost goes without saying that the same methods can be applied when (2.5) is strengthened to (3.8), see Theorem 3. That is, the condition $\sigma(t) \geq 1$ used there can clearly be replaced by the mean value criterion (7.1).

§8. Concluding remarks.

There are two further results closely related to Theorem 2, which are worth adding here.

THEOREM 4. *Theorem 2 continues to hold when conditions* (3.4) *and* (V_2) *are dropped, provided* (3.6) *is replaced by* (3.2)*.*

THEOREM 5. *Theorem 2 continues to hold when conditions* (3.5) *and* (V_3) *are dropped, provided* (3.7) *is replaced by* (3.3)*.*

These results are easily proved by appropriately combining the estimates *6* and *7* given in the proofs of Theorems 1 and 2.

Condition (H_4) can be slightly weakened to the form

$$(Q(t,u,p),u) \leq \delta(t)\,\rho(p) + \delta_1(t) \tag{8.1}$$

(or $(Q(t,u,p),u) \leq \delta(t)\,|p|^{\mu} + \delta_1(t)$ if (3.5) holds). The previous proofs remain essentially unchanged, except for the addition of a term $\omega\,\delta_1 k$ on the right hand side of (6.4), and a term $\alpha\,\omega\,\delta_1 k$ on the right hand side of (6.9). The earlier theorems consequently continue to apply provided that δ_1 satisfies the relation

$$\int_T^t \delta_1(s)k(s)\exp\left(-\int_s^t k(r)\,dr\right)\,ds \;\to\; 0 \qquad \text{as } t \to \infty.$$

A condition of the form (8.1) was also discussed by Murakami [14, Section 4].

If $I = J$ in (H_4), then (3.2) can be slightly improved, namely by requiring only that

$$k = k_1 + k_2,$$

where $k_1 \geq 0$, $k_2 > 0$ and

$$k_1 \in CBV(J), \quad \log k_2 \in \mathrm{Lip}(J).$$

The proof is essentially the same as for Theorem 1. Similar comments apply for Theorems 2 and 3, that is (3.6) and (3.9) can be replaced by

$$k = k_1 + k_3,$$

where $k_1 \geq 0$, $k_3 \geq 0$, $k_1 \in CBV(J)$ and k_3 satisfies (3.6) or (3.9), respectively.

The smoothness condition $F \in C^1(J \times \mathbb{R}^N; \mathbb{R})$ can be weakened to $F \in C(J \times \mathbb{R}^N; \mathbb{R})$ with F_t *and* $\nabla_u F$ *continuous in* $J \times (\mathbb{R}^N \setminus \{0\})$, provided we consider only solutions u such that $u(t) \neq 0$ on J. In the scalar case this means that we treat only solutions of one sign, and in the vector case only solutions whose trajectories never pass through the origin.

A simple example of this situation occurs when $F(t, u) = 2|u|^{1/2}$, $f(t, u) = |u|^{-3/2}u$. Here $(f(t, u), u) = |u|^{1/2}$, and $F_t(t, u) = 0$, so condition (H_2) continues to hold.

When there is no damping in the Lagrangian system (2.1), that is when $Q \equiv 0$, it is well–known that the system can be written in Hamiltonian form (see [20])

$$\begin{cases} u' = \quad \nabla_v \hat{H}(t, u, v) \\ v' = -\nabla_u \hat{H}(t, u, v), \end{cases}$$

where $\hat{H}(t, u, v) = H(u, p) + F(t, u)$, $v = \nabla G(u, p)$. For the quasi–variational system (2.1) the same calculations yield

$$\begin{cases} u' = \quad \nabla_v \hat{H}(t, u, v) \\ v' = -\nabla_u \hat{H}(t, u, v) + \hat{Q}(t, u, v), \end{cases}$$

where now $\hat{Q}(t, u, v) = Q(t, u, p)$.

One may in fact, more generally, consider systems of the form

$$\begin{cases} u' = \quad \nabla_v \hat{H}(t, u, v) + \hat{P}(t, u, v) \\ v' = -\nabla_u \hat{H}(t, u, v) + \hat{Q}(t, u, v), \end{cases} \tag{8.2}$$

and under appropriate conditions seek the asymptotic stability of bounded solutions. In this regard, corresponding to the principal identity (6.1) for (2.1), we have for (8.2) the analogous identity

$$\begin{aligned} \{\omega[\hat{H} + \alpha k(u, v)]\}' &= \omega \hat{H}_t + \omega' \hat{H} + \alpha\omega'[(\nabla_v \hat{H}, v) - (\nabla_u \hat{H}, u)] \\ &\quad + \omega[(\nabla_u \hat{H}, \hat{P}) + (\nabla_v \hat{H}, \hat{Q})] + \alpha\omega'[(\hat{P}, v) + (\hat{Q}, u)] + \alpha\omega''(u, v), \end{aligned}$$

see [11], [16].

Some final comments may be added for the case $N = 1$, for which the result of Lemma 5.5 has the immediate physical interpretation that if the system is *overdamped* (that is, if the damping is so strong that the solution *need not* approach 0 as $t \to \infty$), then necessarily for an overdamped solution the function $g(u) = \frac{1}{2}u^2$ tends to a positive limit and $[g(u)]' = uu'$ is ultimately negative and tends to zero (see (5.11) and recall that $C = 0$ when $N = 1$).

Expressed otherwise, this says that an overdamped solution u necessarily approaches a non–zero limit l as $t \to \infty$ and that

$$u \searrow l \qquad \text{if} \quad l > 0 \qquad \text{and} \qquad u \nearrow l \qquad \text{if} \quad l < 0.$$

A physical interpretation of this result occurs for the damped oscillating pendulum,

$$u'' + a(t)u' + g_e \sin u = 0 \qquad (u < \pi/2)\,,$$

where $a(t) \geq a_0 > 0$ for all t, and g_e denotes the terrestrial gravity. Then either the damping causes the oscillations of the pendulum to decay to zero as $t \to \infty$, or else the ultimate behavior (after a possible final oscillation) is for the pendulum to move with steadily decreasing deflection amplitude to a limiting position *away from equilibrium*. In the latter case the damping is simply so strong that the pendulum, while starting back toward equilibrium, ultimately cannot reach that point. In more colorful terms, this can occur when the pendulum is immersed in a bath of viscous oil, with ever stronger viscosity. An explicit example of this sort was given originally by Artstein & Infante.

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