Global Nonexistence for Abstract Evolution Equations with Positive Initial Energy

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Abstract. In this paper we consider the problem of non-continuation of solutions of the initial value problem for abstract evolution equations of the form

$$Pu_{tt} + Q(t)u_t + A(t,u) = F(t,u), \quad t \in J = [0,\infty)$$

where $P =$ and Q are linear self-adjoint operators, and $A(t,u)$ and $F(t,u)$ are respectively a linear operator in u (typically of differential type) and a nonlinear driving force. Our principal concern is the non-continuation (or blow–up) of solutions when the initial energy is positive, but appropriately bounded.

§1. Introduction.

In a recent paper [7] the problem of non–continuation was studied for abstract evolution equations of the type

$$Pu_{tt} + Q(t)u_t + A(t,u) = F(t,u), \quad t \in J = [0,\infty), \tag{1.1}$$

where P and $Q(t)$ are linear self-adjoint operators, and $A(t,u)$ and $F(t,u)$ are typically a divergence operator in u and a nonlinear driving force.

Other versions of (1.1) were considered earlier by Levine [3–6], for which he introduced the important technique of "concavity" analysis of auxiliary second order differential inequalities. In all these papers the principal mechanism of blow–up was the assumption of negative initial energy.

In an interesting paper [10], which has just appeared, Ono has also used concavity analysis to study blow–up, but in the more general case when the initial energy is allowed to take appropriately small positive values. His analysis primarily considers linear wave operators, and moreover is restricted to bounded domains in $\mathbb{R}^n$. (It should, however, be added that Ono also allows Kirchhoff type operators, an added generalization but without serious affect on the principal ideas.)

It is the purpose of this paper to extend Ono's analysis to the abstract equation (1.1), which we do in Theorem 1. Moreover, in concrete cases, we introduce appropriate methods to treat divergence structure operators in unbounded domains (including but not

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necessarily restricted to $\mathbb{R}^n$). Our conclusions also yield a larger class of initial data than in [10] for which non-continuation (or blow–up) must occur; see Remark 1 in Section 3.

In the next section we give a precise meaning to equation (1.1), and give our main abstract theorem. Section 3 discusses a divergence structure equation in $\mathbb{R}^n$ for which non-continuation occurs for positive initial energy, even for unbounded domains. Here the primary new idea, in comparison with [7] and [10], is to introduce an appropriate coercive operator associated with the equation.

§2. The main theorem.

Let X be a Banach space, and X' its dual space. If $x \in X$ and $x' \in X'$, we shall write $\langle x', x\rangle_X$ to denote the natural pairing of x and x', that is $\langle x', x\rangle_X = x'(x)$.

Let V be a Hilbert space. An operator $P : V \to V'$ will be called *symmetric* if

$$\langle Pv, w\rangle_V = \langle Pw, v\rangle_V \quad \text{for all } v, w \in V,$$

and *non–negative definite* if

$$\langle Pv, v\rangle_V \geq 0 \quad \text{for all } v \in V.$$

It is easy to check that a symmetric operator must be linear and, moreover, continuous by the uniform boundedness theorem.

We consider the evolution equation (1.1), where P is symmetric and non–negative definite from V into V'. We suppose that the *dissipation operator* $Q(t)$ is, for each $t \in J$, symmetric and non–negative definite from an appropriate Hilbert space Y into its dual Y'. In addition, assume $Q \in C(J \to B(Y, Y'))$, that is $\langle Q(\cdot)v, w\rangle_Y : J \to \mathbb{R}$ is continuous for each $v, w \in Y$. Note that $P \equiv 0$ and $Q \equiv 0$ are specifically allowed.

Finally, the operators A and F are such that[2]

$$A : J \times W \to W', \qquad F : J \times X \to X',$$

with W, X Banach spaces and W', X' their duals. In order to define the energy $\mathcal{E}u$ of a solution of (1.1), see below, it is necessary that there exist C^1 potentials

$$\mathcal{A} : J \times W \to \mathbb{R}, \qquad \mathcal{F} : J \times X \to \mathbb{R},$$

such that for each fixed t the operators A and F are the Fréchet derivatives with respect to u of $\mathcal{A}$ and $\mathcal{F}$, respectively; by normalization we can take $\mathcal{A}(t, 0) \equiv 0$, $\mathcal{F}(t, 0) \equiv 0$.

Now suppose that there is given a nontrivial subspace G of V, W, X and Y – not necessarily closed. Let

$$K = \{\varphi : J \to G \mid \varphi \in C(J \to W) \cap C(J \to X) \cap C^1(J \to V) \cap AC(J \to Y)\}.$$

[2] Specific examples are given in [4, 7, 8, 11], and also in Section 3 below. For further clarity and definiteness we refer the reader to these papers.

We say that u is a (*strong*) *solution* of (1.1) if

(a) $u \in K$;

(b) Distribution Identity:

$$\langle Pu_t(\tau), \varphi(\tau)\rangle_V \Big|_0^t = \int_0^t \{\langle Pu_t(\tau), \varphi_t(\tau)\rangle_V - \langle Q(\tau)u_t(\tau), \varphi(\tau)\rangle_Y - \langle A(\tau, u(\tau)), \varphi(\tau)\rangle_W + \langle F(\tau, u(\tau)), \varphi(\tau)\rangle_X\} d\tau$$

for all $t \in J$ and $\varphi \in K$;

(c) Energy Conservation:

$$\mathcal{E}u(t) - \mathcal{E}u(0) \le -\int_0^t \{\langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y - \mathcal{A}_t(\tau, u(\tau)) + \mathcal{F}_t(\tau, u(\tau))\} d\tau, \tag{2.1}$$

where

$$\mathcal{E}u(t) = \tfrac{1}{2}\langle Pu_t(t), u_t(t)\rangle_V + \mathcal{A}(t, u(t)) - \mathcal{F}(t, u(t)), \quad t \in J, \tag{2.2}$$

is the total energy of u.

Assume even more that $Q \in C^1(J \to B(Y, Y'))$, with $Q_t(t) : Y \to Y'$ being non–positive definite and (necessarily) symmetric for all $t \in J$.

Suppose there are constants $p \ge q$ such that, for all $(t, u) \in J \times G$,

$$\langle A(t, u), u\rangle_W - \langle F(t, u), u\rangle_X \le q\mathcal{A}(t, u) - p\mathcal{F}(t, u) \tag{2.3}$$

and

$$\mathcal{A}_t(t, u) - \mathcal{F}_t(t, u) \le 0. \tag{2.4}$$

Theorem 1. *Assume that* (2.3) *and* (2.4) *hold.*

(i) *Let $p > 2$. Then there is no solution $u = u(t)$ of* (1.1) *on J with*

$$\mathcal{A}(t, u(t)) \ge \lambda_0 > 0, \qquad t \in J, \tag{2.5}$$

and

$$\mathcal{E}u(0) < \left(1 - \frac{q}{p}\right)\lambda_0 = D_0.$$

(ii) *Let $q > 2$. Then there is no solution $u = u(t)$ of* (1.1) *on J with*

$$\mathcal{F}(t, u(t)) \ge \lambda_1 > 0, \qquad t \in J, \tag{2.6}$$

and

$$\mathcal{E}u(0) < \left(\frac{p}{q} - 1\right)\lambda_1 = D_1.$$

Proof. We define, corresponding to any solution u of (1.1) on J,

$$\begin{aligned}\mathcal{I}(t) = \langle Pu(t), u(t)\rangle_V + \int_0^t \{\langle Q(\tau)u(\tau), u(\tau)\rangle_Y + (\tau - t)\langle Q_t(\tau)u(\tau), u(\tau)\rangle_Y\}d\tau \\ + (T_0 - t)\langle Q(0)u(0), u(0)\rangle_Y + \beta(t + t_0)^2,\end{aligned} \tag{2.7}$$

where t_0, T_0, β are positive constants which will be fixed later (see Levine [3–5], and also [7]). Then one finds, from the assumption that P, $Q(t)$, $Q_t(t)$ are linear, continuous and symmetric,

$$\begin{aligned}\mathcal{I}'(t) &= 2\langle Pu(t), u_t(t)\rangle_V + \langle Q(t)u(t), u(t)\rangle_Y - \langle Q(0)u(0), u(0)\rangle_Y \\ &\quad - \int_0^t \langle Q_t(\tau)u(\tau), u(\tau)\rangle_Y + 2\beta(t + t_0) \\ &= 2\langle Pu(t), u_t(t)\rangle_V + 2\int_0^t \langle Q(\tau)u(\tau), u_t(\tau)\rangle_Y d\tau + 2\beta(t + t_0).\end{aligned}$$

¿From the distribution identity (b), by taking $\varphi = u \in K$ it follows next that

$$\begin{aligned}\tfrac{1}{2}\mathcal{I}''(t) &= \{\langle Pu_t(t), u_t(t)\rangle_V - \langle Q(t)u_t(t), u(t)\rangle_Y - \langle A(t, u(t)), u(t)\rangle_W + \langle F(t, u(t)), u(t)\rangle_X\} \\ &\qquad + \langle Q(t)u(t), u_t(t)\rangle_Y + \beta \\ &= \langle Pu_t(t), u_t(t)\rangle_V - \langle A(t, u(t)), u(t)\rangle_W + \langle F(t, u(t)), u(t)\rangle_X + \beta.\end{aligned}$$

This may be simplified by using (2.3) and (2.2), namely

$$\begin{aligned}\tfrac{1}{2}\mathcal{I}''(t) &\geq \langle Pu_t(t), u_t(t)\rangle_V - q\mathcal{A}(t, u(t)) + p\mathcal{F}(t, u(t)) + \beta \\ &= \langle Pu_t(t), u_t(t)_V\rangle - q\mathcal{A}(t, u(t)) + p\left(\tfrac{1}{2}\langle Pu_t(t), u_t(t)\rangle_V + \mathcal{A}(t, u(t)) - \mathcal{E}u(t)\right) + \beta \\ &= \left(1 + \frac{p}{2}\right)\langle Pu_t(t), u_t(t)_V\rangle + (p - q)\mathcal{A}(t, u(t)) - p\mathcal{E}u(t) + \beta.\end{aligned}$$

Now $(p - q)\mathcal{A}(t, u(t)) \geq (p - q)\lambda_0 = pD_0$ by the hypothesis (2.5), and also

$$\mathcal{E}u(t) \leq \mathcal{E}u(0) - \int_0^t \langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y d\tau$$

by (2.1) and (2.4). Therefore we find

$$\tfrac{1}{2}\mathcal{I}''(t) \geq \left(1 + \frac{p}{2}\right)\langle Pu_t(t), u_t(t)_V\rangle + p\int_0^t \langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y d\tau + p\{D_0 - \mathcal{E}u(0)\} + \beta.$$

Let $\beta = 2\{D_0 - \mathcal{E}u(0)\} > 0$. This gives the main estimate

$$\mathcal{I}''(t) \geq (p+2)\{\langle Pu_t(t), u_t(t)\rangle_V + \beta\} + 2p\int_0^t \langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y d\tau. \tag{2.8}$$

The proof of part (i) is now almost exactly the same as in [7], from their formula (2.8) onward. For the convenience of the reader we include it here.

Take t_0 so large that $\mathcal{I}'(0) = 2\langle Pu(0), u_t(0)\rangle_V + 2\beta t_0 > 0$. Then, using the fact that P, $Q(t)$ are non–negative definite there results

$$\mathcal{I}'',\ \mathcal{I}',\ \mathcal{I} > 0 \qquad \text{on } J.$$

We assert that

$$\mathcal{I}\mathcal{I}'' - \alpha\mathcal{I}'^2 \geq 0 \quad \text{on } [0, T_0], \tag{2.9}$$

where $\alpha = (p+2)/4$. Indeed, put

$$\mathbb{A} = \langle Pu(t), u(t)\rangle_V + \int_0^t \langle Q(\tau)u(\tau), u(\tau)\rangle_Y d\tau + \beta(t+t_0)^2,$$
$$\mathbb{C} = \langle Pu_t(t), u_t(t)\rangle_V + \int_0^t \langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y d\tau + \beta,$$

and $\mathbb{B} = \frac{1}{2}\mathcal{I}'$. Since $Q(t)$ is non–negative definite and $Q_t(t)$ non–positive definite for each $t \in J$, we see that

$$\mathbb{A} \leq \mathcal{I} \qquad \text{on } [0, T_0]. \tag{2.10}$$

Moreover, by (2.8) and the fact that $2p > p+2$,

$$\mathbb{C} \leq \mathcal{I}''/(p+2) \qquad \text{on } J. \tag{2.11}$$

Now observe that, for all $(\xi, \eta) \in \mathbb{R}^2$ and $t \in J$,

$$\begin{aligned}\mathbb{A}\xi^2 + 2\mathbb{B}\xi\eta + \mathbb{C}\eta^2 = {} & \langle \xi Pu(t) + \eta Pu_t(t), \xi u(t) + \eta u_t(t)\rangle_V \\ & + \int_0^t \langle \xi Q(\tau)u(\tau) + \eta Q(\tau)u_t(\tau), \xi u(\tau) + \eta u_t(\tau)\rangle_Y d\tau \\ & + \beta\{(t+t_0)\xi + \eta\}^2 \geq 0,\end{aligned}$$

because P, $Q(t)$ are linear, symmetric and non–negative definite. Thus $\mathbb{AC} - \mathbb{B}^2 \geq 0$. In turn (2.9) is valid by virtue of (2.10), (2.11) and the fact that $\mathbb{A}$, $\mathbb{C} > 0$.

Of course $\alpha > 1$ since $p > 2$ by assumption. The inequality (2.9) can be written as $(\mathcal{I}^{-\alpha}\mathcal{I}')' \geq 0$, so

$$\frac{\mathcal{I}'(t)}{\mathcal{I}^\alpha(t)} \geq \frac{\mathcal{I}'(0)}{\mathcal{I}^\alpha(0)} > 0 \quad \text{for } t \in [0, T_0].$$

This is a Riccati inequality with blow–up time

$$T < \frac{1}{\alpha - 1}\frac{\mathcal{I}(0)}{\mathcal{I}'(0)}.$$

Consequently, if T_0 is chosen as the right hand side of the above inequality, we have a contradiction. In fact, since $\mathcal{I}(0)$ depends on T_0, this gives an easily solved linear equation for T_0, the solution being *positive* for all t_0 large enough, e.g., whenever

$$\beta t_0 > \frac{2}{p-2}\langle Q(0)u(0), u(0)\rangle_Y - \langle Pu(0), u_t(0)\rangle_V.$$

(An optimal choice for t_0, to minimize T_0 and to provide a specific estimate for the blow–up time, is easily determined, but is unnecessary for our purposes.) This completes the proof of part (i) of the theorem.

For the proof of part (ii) we proceed almost exactly as in part (i), obtaining in place of (2.8) the estimate

$$\tfrac{1}{2}\mathcal{I}''(t) \geq \left(1 + \frac{q}{2}\right)\langle Pu_t(t), u_t(t)\rangle_V + q\int_0^t \langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y d\tau + q\{D_1 - \mathcal{E}u(0)\} + \beta.$$

Now set $\beta = 2\{D_1 - \mathcal{E}u(0)\} > 0$, and from here on the proof is the same as in part (i).

Remarks. 1. If $p = q$ in (2.3), then we see from the proof that the results (i) and (ii) remain valid with $D_0 = D_1 = 0$, but without requiring either (2.5) or (2.6). In other words, non–continuation holds under the single condition $\mathcal{E}u(0) < 0$, namely, negative initial energy; this is exactly the main result of [7].

2. In the usual applications $\mathcal{A}$ is independent of t, in which case (2.4) reduces simply to $\mathcal{F}_t(t, u) \geq 0$ on $J \times G$.

3. In recent work [13] Vitillaro has treated positive energy blow–up for abstract evolution equations of the type

$$[P(u_t)]_t + Q(t, u_t) + A(u) = F(u), \quad t \in J = [0, \infty),$$

where the damping term Q is allowed to be nonlinear in u_t. To compensate for this increased generality, however, he requires stronger conditions on $A(u)$ and $F(u)$ than purely the assumption (2.3).

§**3. Examples.**

Let Ω be an open domain in $\mathbb{R}^n$ and consider the model problem

$$u_{tt} - \operatorname{div}(|Du|^{q-2}Du) + \mu|u|^{q-2}u = f(t,x,u), \quad x \in \Omega,\ t \in J, \tag{3.1}$$

where

$$f(t,x,u) = g(t,x)|u|^{\sigma-2}u + c|u|^{p-2}u, \tag{3.2}$$

and

$$\mu \geq 0, \quad 1 < q < p; \qquad c > 0, \quad 1 < \sigma < p. \tag{3.3}$$

For the function g we assume

$$-g,\ \frac{\partial g}{\partial t} \geq 0 \quad \text{on } J \times \Omega, \qquad g(t,\cdot) \in L^{p/(p-\sigma)}(\Omega) \quad \text{for all } t \in J. \tag{3.4}$$

Here the appropriate spaces are $V = L^2(\Omega)$, $W = W_0^{1,q}(\Omega)$, $X = L^p(\Omega)$ and $G = L^2(\Omega) \cap L^p(\Omega) \cap W_0^{1,q}(\Omega)$. For definiteness the space W will be endowed with the norm

$$\|u\|_W = (\|u\|^q_{L^q(\Omega)} + \|Du\|^q_{L^q(\Omega)})^{1/q}.$$

(Note that the remaining space Y is unneeded, since for simplicity we have omitted damping terms from the equation. In fact, adding a term $a(t,x)u_t$, $a \geq 0$, $\partial a/\partial t \leq 0$ on $J \times \Omega$, to the left hand side of (3.1) leaves Theorems 2 and 3 below unchanged.)

The operator P corresponding to (3.1) is given by $\langle Pv, w\rangle_V = (v,w)_{L^2}$; clearly P is symmetric and positive definite. We take $A(u) = -\operatorname{div}(|Du|^{q-2}Du) + \mu|u|^{q-2}u$, so that[3]

$$\langle A(u), u\rangle_W = \|Du\|^q_{L^q} + \mu\|u\|^q_{L^q} = q\mathcal{A}(u).$$

On the other hand, as is easy to see, one must then have

$$\mathcal{F}(t,u) = \frac{1}{\sigma}\int_\Omega g(t,x)|u|^\sigma dx + \frac{c}{p}\|u\|^p_{L^p}. \tag{3.5}$$

By a solution of (3.1) we now mean a solution of the abstract evolution equation (1.1) corresponding to the operators P, A and F just defined.

Clearly (2.4) is satisfied on $J \times G$ by $(3.4)_1$ – we assume suitable regularity of $g = g(t,x)$ so that $\mathcal{F}_t(t,u)$ can be calculated on $J \times G$ by differentiation under the integral sign in (3.5). Finally,

$$\langle F(t,u), u\rangle_X = \int_\Omega g(t,x)|u|^\sigma dx + c\|u\|^p_{L^p},$$

[3] The choice $A(u) = -\operatorname{div}(|Du|^{q-2}Du)$ is inconvenient when $\Omega = \mathbb{R}^n$, or indeed when Ω has infinite measure, due to the lack of coercivity.

so $p\mathcal{F}(t,u) \ge \langle F(t,u), u\rangle_X$ because $\sigma < p$ and $g \le 0$. Therefore (2.3) is verified on $J \times G$.

In order to apply Theorem 1 we shall verify that condition (2.5) holds for appropriate initial data. The situation is different when Ω is bounded, and for Ω unbounded. For definiteness we treat the (somewhat more difficult) unbounded case, assuming the further restrictions

$$\mu > 0, \qquad q < p \le r, \tag{3.6}$$

where $r = nq/(n-q)$ is the Sobolev exponent for $W_0^{1,q}(\Omega)$ when $q < n$, or otherwise $q < p < \infty$ if $q \ge n$.

It is well known that there is a constant $C > 0$ depending only on n, p and q such that

$$\|u\|_{L^p} \le C\|u\|_W \qquad \text{for all } u \in W,$$

see Adams [1, Theorem 5.4, Part III, pp. 97–98]. It follows that

$$\mathcal{A}(u) \ge B\|u\|^q_{L^p}, \qquad u \in W, \tag{3.7}$$

for $B = \min\{1,\ \mu\}/qC^q$. For precision, we take B in fact to be the *supremum* of values B for which the coercivity condition (3.7) holds (that (3.7) is actually satisfied for the supremum value of B can be proved, but in any case the final results are independent of this).

Using (3.7), it now follows from (2.2), (2.4) and $(3.4)_1$ that

$$\mathcal{E}u(t) \ge \mathcal{A}(u) - \mathcal{F}(t,u) \ge B\|u\|^q_{L^p} - \frac{c}{p}\|u\|^p_{L^p} = E(\lambda), \tag{3.8}$$

where $E(\lambda) = B\lambda^q - (c/p)\lambda^p$ and $\lambda = \lambda(t) = \|u\|_{L^p}$.

The graph of the function $E(\lambda) = B\lambda^q - (c/p)\lambda^p$ on $\lambda \ge 0$ has a single maximum value E_0 at $\lambda = \Lambda_0$, where

$$\Lambda_0^{p-q} = \frac{qB}{c}, \qquad E_0 = \left(\frac{q}{c}\right)^{q/(p-q)} B^{p/(p-q)}\left(1 - \frac{q}{p}\right). \tag{3.9}$$

Suppose the initial data is such that $\mathcal{E}u(0)$, $\lambda(0)$ satisfies

$$\mathcal{E}u(0) < E_0, \qquad \lambda(0) > \Lambda_0. \tag{3.10}$$

Then clearly from (2.1) and (2.4) we have $\mathcal{E}u(t) < E_0$ for all $t > 0$. At the same time, by (3.8) it is evident that there can be no time $t > 0$ for which $\mathcal{E}u(t) < E_0$ and $\lambda(t) = \Lambda_0$. Hence by continuity of $\mathcal{E}u$ and λ also we have $\lambda(t) > \Lambda_0$ for all $t > 0$. This in turn implies by (3.7) that

$$\mathcal{A}(u(t)) \ge B\|u(t)\|^q_{L^p} > B\Lambda_0 \equiv \lambda_0,$$

proving that (2.5) holds when (3.10) is satisfied. This proves our second main theorem.

Theorem 2. *Under the conditions* (3.3)*,* (3.4)*,* (3.6)*, and the further restriction* $p > 2$*, the problem* (3.1)*,* (3.2) *cannot have any global solution* u *corresponding to initial data*

$$\mathcal{E}u(0) < E_0, \qquad \|u(0)\|_{L^p} > \Lambda_0, \tag{3.11}$$

where E_0*,* Λ_0 *are given by* (3.9) *and* B *is the best constant for the coercive potential* $\mathcal{A}(u)$*, see* (3.7).

The corresponding result for bounded domains Ω is slightly different.

Theorem 3. *Let the measure of* Ω *be finite. Then Theorem 2 remains valid even when* $\mu = 0$.

To see this, it is enough to note that for domains having finite measure we can use for W the equivalent norm $\|u\|_W = \|Du\|_{L^q}$.

Remarks. 1. Ono in [10] essentially treats the semilinear case $q = 2$, $\mu = 0$ of Theorem 3, but with the initial data satisfying the somewhat stronger conditions

$$\mathcal{E}u(0) < E_0, \qquad \|Du(0)\|_{L^2}^2 < c\|u(0)\|_{L^p}^p.$$

Other work involving positive initial energies appears earlier in [2] and [9], the first however restricted to the wave operator itself (with nonlinear boundary conditions), and the second with less precise bounds for the initial energy.

2. In general, (2.6) cannot be obtained when $\mathcal{E}u(0) > 0$. This can be seen, for example, when the function $-g$ is sufficiently large.

3. The discussion above makes clear the distinction between the case when $\mathcal{E}u(0)$ is taken to be negative, and when it is allowed to be positive, as in (3.11). In particular, in the latter case it is necessary that the potential $\mathcal{A}(u)$ be coercive so that in turn one must assume that $p \le r$, a condition which was not needed in the corresponding examples in Section 4 of [7].

4. In the example concerning the degenerate s–Laplacian on page 262 of [7] the condition $s > 2$ was required for the application of their Theorem 1. Here we assume only that $s = q > 1$. This significant improvement is made possible by the more general form of condition (2.3) here in comparison with the corresponding assumption of [7].

5. The results of Theorems 2 and 3 arise directly from the algebraic behavior of the function $E(\lambda)$ representing the potential well for (3.1). For initial data which lies deep enough in the "well" itself, the corresponding solutions are asymptotically stable. This dichotomy is discussed in detail in [12].

A number of concrete examples relative to linear operators A were given in Section III of [4], to which we refer the reader. Example VI of [4, p.16] in particular deserves special mention. Here the operator $-Q$ is the Laplacian, so for precision the space Y as well as W must be chosen as $H_0^1(\Omega)$.

Other concrete operators $A(u)$ are given in Section 6 of [11], notably the polyharmonic operator $(-\Delta)^L$, where $L \geq 1$ is an integer, and still further examples are given in Section 4 of [8].

All of these examples allow extensions to the time dependent case and also exhibit blow–up for positive initial energies, both for bounded and unbounded domains, as discussed above.

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