Archive Rat. mech. Anal., (1995).

ASYMPTOTIC STABILITY FOR ORDINARY DIFFERENTIAL SYSTEMS WITH TIME DEPENDENT RESTORING POTENTIALS

by

PATRIZIA PUCCI
Università di Perugia
Perugia, Italy

and

JAMES SERRIN
University of Minnesota
Minneapolis, MN, U.S.A.

1. Introduction

Asymptotic stability for the classical second order ordinary differential equation

$$u'' + q^2(t)u = 0, \qquad t \in [T, \infty), \tag{1.1}$$

where $q(t) > 0$, has been widely studied in the literature. Important results are due to Sansone, Wiman and Ballieu & Peiffer, and a good bibliography is given by Cesari.

In particular it was shown by Ballieu & Peiffer that if there exists a non-increasing continuous function k such that

(i) $$q'(t) \geq k(t)q^2(t) > 0 \qquad \text{for } t \in [T, \infty),$$

(ii) $$\int^{\infty} k(t)q(t)dt = \infty,$$

(iii) $$q'/q^2 \text{ is bounded in } [T, \infty),$$

then every solution u of (1.1) tends to zero as $t \to \infty$. Sansone earlier had obtained the special case $k = \text{Const.}/q^2$, but without requiring condition (iii).

Here, as a consequence of our general conclusions we establish the following theorem, containing both the results of Sansone [14, page 67] and of Ballieu & Peiffer [1, Corollary 11].

Theorem A. *Suppose that there exists a non-negative continuous function k of bounded variation on $[T,\infty)$, such that*

$$\text{(i)} \qquad q'(t) \geq k(t)q^2(t) \qquad \textit{for } t \in [T,\infty);$$

$$\text{(ii)} \qquad \int^{\infty} k(t)q(t)dt = \infty;$$

$$\text{(iii)} \qquad \liminf_{t\to\infty} \int^t k^2(s)\frac{q'(s)}{q(s)}\,ds \bigg/ \int^t k(s)q(s)ds < \infty.$$

Then every solution of (1.1) *tends to zero as $t \to \infty$.*

The cases $q(t) = t^\beta$ and $q(t) = (\log t)^\beta$, $\beta > 0$, are covered by Theorem A as well as by Ballieu & Peiffer, if we take respectively $k(t) = \beta t^{-\beta-1}$ and $k(t) = \beta/t(\log t)^{\beta+1}$. Condition (iii) of Theorem A is implied by (iii) of the theorem of Ballieu & Peiffer, and even by the weaker pointwise condition, kq'/q^2 bounded on $[T,\infty)$. The hypotheses (i) and (ii), on the other hand, show that $q'/q \notin L^1[T,\infty)$. This is in fact a *necessary condition* for $u = 0$ to be an attractor as $t \to \infty$, as follows by applying Theorem 5 of [10] to the transformed system (2.9).

By taking $k(t) = \text{Const.}/tq(t)$ we obtain the following striking, and apparently new, result. (In this case k is clearly continuous and of bounded variation on $[T,\infty)$ since q is certainly non-decreasing.)

Corollary A1. *Suppose that*

$$tq'(t) \geq \text{Const.}\, q(t) \qquad \textit{for } t \in [T,\infty).$$

Then every solution of (1.1) *tends to zero as $t \to \infty$.*

A variant of Corollary A1 is the following slightly more general result.

Corollary A2. *Suppose that there exists a non-negative continuous function ψ of bounded variation on $[T,\infty)$, such that*

$$q' \geq \psi\, q \quad \textit{in } [T,\infty), \qquad \int^{\infty} \psi(t)dt = \infty.$$

Then every solution of (1.1) *tends to zero as $t \to \infty$.*

The important advantage of Corollary A2 over the result of Ballieu & Peiffer is that no upper bound on the growth rate of q is required. The theorem of Sansone, moreover, arises as the special case $\psi(t) = \text{Const.}/q(t)$.

In all the above results necessarily $q(t) \nearrow \infty$ as $t \to \infty$, so that under the given assumptions all solutions are oscillatory with frequency approaching ∞ and amplitude approaching zero as $t \to \infty$. The case $q(t) \nearrow \infty$ as $t \to \infty$ was treated also by Prodi, who showed under this assumption alone that there exists *at least one solution* of (1.1) which tends to zero as $t \to \infty$. Prodi's result, like Corollary A2, places no upper bound on the growth rate of q; the advantage of Corollary A2 is that *every solution* must tend to zero as $t \to \infty$, though of course this is bought at the expense of the stronger condition $q' \geq \psi\, q$.

Consider now the equation

$$u'' + h(t)u' + q^2(t)f(u) = 0, \qquad t \in [T, \infty), \tag{1.2}$$

where f, $h : \mathbb{R} \to \mathbb{R}$ are continuous functions, with f satisfying the restoring condition

$$uf(u) > 0 \qquad \text{for } u \neq 0. \tag{1.3}$$

Equation (1.2), which has similar but more delicate behavior than (1.1), has been studied by Salvadori and, in the linear case, by Ballieu & Peiffer and by Corne. Here we have the following conclusion.

Theorem B. *Let $A = h + q'/q$. Suppose that there exists a non-negative continuous function k of bounded variation on $[T, \infty)$, such that*

(i) $$A \geq kq \qquad \text{in } [T, \infty);$$

(ii) $$\int^{\infty} k(t)q(t)dt = \infty;$$

(iii) $$\liminf_{t\to\infty} \int^t k^2(s)A(s)ds \Big/ \int^t k(s)q(s)ds < \infty.$$

Then every bounded solution of (1.2) *tends to zero as $t \to \infty$.*

If $f^+, f^- \notin L^1(\mathbb{R})$ then the conclusion holds for all solutions of (1.2).

Theorem A is the special case of Theorem B when $h = 0$ and $f(u) = u$. Theorem B also includes Corollary 8 of [1] by taking $k = 1/F$, in their notation.

An important example of (1.2) is the radial Matukuma equation,

$$u'' + \frac{n-1}{t}\,u' + \frac{u|u|^{P-1}}{1+t^2} = 0, \qquad n > 2,\ P > 1. \tag{1.4}$$

It is easy to see that the hypotheses of Theorem B are satisfied with $k(t) = \rho$, where $\rho \in (0, n-2)$. Hence *all solutions of* (1.4) *approach zero as* $t \to \infty$. For positive solutions on $[0, \infty)$ of (1.4) this is a well-known result, see for example [6], [7] and the references quoted therein.

If the term $1 + t^2$ in (1.4) is replaced by $(1 + t^2)^\gamma$, then by the same proof solutions necessarily approach zero when $\gamma \le 1$ (take $k(t) = \rho\, t^{\gamma-1}$). On the other hand, if $\gamma > 1$ there exist solutions which tend to finite non-zero limits as $t \to \infty$. Thus the Matukuma equation is a borderline case for asymptotic stability of the rest state. In fact a more precise borderline is provided by the function $(1 + t^2)\log(1 + t^2)$, as will be seen in Theorems 7 and 8 of Section 4.2.

Finally, Theorem B has the following corollaries corresponding to those of Theorem A. (Throughout the paper, the notation Const. denotes a generic positive constant.)

Corollary B1. *Let* $h \in L^1[T, \infty)$*. Suppose that*

$$tA(t) \ge \text{Const.} \qquad \textit{for } t \in [T, \infty).$$

Then every bounded solution of (1.2) *tends to zero as* $t \to \infty$.

Corollary B2. *Let* $h \in L^1[T, \infty)$*. Assume that there exists a non-negative continuous function* ψ *of bounded variation on* $[T, \infty)$*, such that*

$$A \ge \psi \quad \textit{in } [T, \infty), \qquad \int^\infty \psi(t)dt = \infty.$$

Then every bounded solution of (1.2) *tends to zero as* $t \to \infty$.

To prove these results, put $d(t) = \exp \int^t h(s)ds$ and observe that $A = (dq)'/dq$, that dq is non-decreasing since $A \ge 0$, and that d is of bounded variation on $[T, \infty)$ since $h \in L^1[T, \infty)$. One can then proceed as for Corollaries A1 and A2.

In both corollaries, if q is non-decreasing (or more generally if $1/q$ is of bounded variation), then the condition $h \in L^1[T,\infty)$ can be replaced by the weaker hypothesis that $\sup\{\int^t h(s)ds < \infty \ : \ t \in [T,\infty)\}$ is finite, since it is no longer necessary to have d of bounded variation. This remark applies in particular when $h \leq 0$, and also, more interestingly, when h takes both positive and negative values, e.g. $h(t) = \sin t/t$.

When $\int^t h(s)ds$ is unbounded above, the situation is somewhat more complicated, see Theorems 5 and 6 below. We give one representative corollary which is applicable in this case.

Corollary B3. *Let $q'(t) \geq 0$ for $t \in [T,\infty)$. Suppose there exists a non-negative continuous function ψ of bounded variation on $[T,\infty)$, such that*

$$A \geq \psi, \qquad \psi\, h \leq \text{Const.}\, q^2 \quad \textit{in } [T,\infty),$$

and $\int^\infty \psi(t)dt = \infty$. Then every bounded solution of (1.2) tends to zero as $t \to \infty$.

We now turn to the main purpose of the paper, to develop a fairly general asymptotic stability theory for ordinary differential systems, containing results of the type discussed above as special cases. We include, as well, nonlinear mechanical systems of the sort introduced by Salvadori [13] and also systems arising from degenerate variational problems.

For the rest of the paper, therefore, we shall consider vector unknowns $u : J \to \mathbb{R}^N$, where J is a half open interval of the type $[T,\infty)$, and systems of the form

$$(\nabla_p G(u,u'))' - \nabla_u G(u,u') + q^m(t) f(u) = Q(t,u,u'), \qquad t \in J. \tag{1.5}$$

The most important assumptions on (1.5) are that $G = G(u,p)$ be strictly convex and homogeneous of degree $m > 1$ in the variable $p \in \mathbb{R}^N$, that $f = \nabla_u F$ be a continuous function of gradient type satisfying $(f(u),u) > 0$ for $u \neq 0$, and that Q be a continuous vector function satisfying the growth conditions (H_4) below.

There are a number of interesting examples which can be represented by the system (1.5). The case studied in [13] is given by $G(u,p) = \frac{1}{2}(M(u)p,p)$ and $Q(t,u,p) =$

$-L(t,u)p$, where L is a continuous, bounded and uniformly positive definite matrix and M is a positive definite matrix of class C^1.

When $G(p) = \sum_{i=1}^N |p_i|^m/m$, $m > 1$, the system (1.5) reduces to

$$(|u_i'|^{m-2}u_i')' + q^m(t)f_i(u) = Q_i(t,u,u'), \qquad i = 1,\dots,N, \tag{1.6}$$

which is singular when $m \neq 2$ at points where at least one $u_i' = 0$. If $m = 2$ the system (1.6) takes the familiar form

$$u'' + q^2(t)f(u) = Q(t,u,u'), \tag{1.7}$$

and, when $Q = 0$,

$$u'' + q^2(t)f(u) = 0. \tag{1.8}$$

For this system, in which of course $(f(u),u) > 0$ for $u \neq 0$ and f is of gradient type, $f = \nabla_u F$, we have the following results corresponding to Theorem A and its corollaries.

Theorem C. *Assume that the hypotheses of Theorem A, or of Corollary A1 or A2, are satisfied. Then every bounded solution of the system* (1.8) *tends to zero as* $t \to \infty$.

If additionally $F(u) \to \infty$ *as* $|u| \to \infty$*, then all solutions of* (1.8) *approach zero as* $t \to \infty$.

A final example worth noting is the Euler-Lagrange system for extremals of the variational integral

$$\int_J g(t)\{G(u,u') - q^m(t)F(u)\}dt,$$

where G, F and q are of the type discussed above and $g : J \to \mathbb{R}$ is a positive function of class C^1. Extremals for this functional satisfy (1.5) with

$$Q(t,u,p) = -\frac{g'(t)}{g(t)}\,\nabla_p G(u,p). \tag{1.9}$$

The paper is organized as follows. Section 2 contains the formulation of the problem and preliminary results, including a boundedness theorem for solutions of (1.5) which is of independent interest. In Section 3 we give our main theorems. Theorem 1 deals with the general system (1.5), while Theorem 2 treats the variational subcase (1.9). Theorems A-C above are special cases of Theorem 2.

Section 4 includes applications to holonomic mechanical systems with N degrees of freedom and strongly time dependent restoring potentials, to generalized Matukuma equations, and to the canonical equation (1.2). These applications – *which are a central part of the paper* – are consequences of the principal Theorems 5-8 of Section 4. Theorems 5 and 6 in particular bear the same relation to Theorem 1 as the corollaries above bear to Theorems A and B. Finally Theorems 7 and 8 are concerned with equation (1.2) in the special case $h(t) = (n-1)/t$, corresponding to the radial Matukuma equation (1.4) and to related problems for partial differential equations.

The last part of Section 4 treats the case when the restoring potential $q^m(t)F(u)$ for (1.5) is replaced by the more general form $\Phi(t,u)$, see (4.31).

2. Formulation of the problem

We consider the system (1.5) under the following specific hypotheses, which are assumed to hold throughout the paper unless otherwise mentioned.

(H_1) $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N; \mathbb{R})$; $\quad G(u,\cdot)$ is strictly convex and homogeneous of degree $m > 1$; for every $U > 0$ and $p_0 > 0$ there is a positive constant such that

$$(\nabla_u G(u,p), u) \le \text{Const.}\, G(u,p) \qquad \text{for all } |u| \le U \text{and } |p| \ge p_0.$$

(H_2) There exists $F \in C^1(\mathbb{R}^N; \mathbb{R})$ such that $f = \nabla_u F$ and

$$(f(u), u) > 0 \qquad \text{for } u \ne 0. \tag{2.1}$$

(H_3)
$$q \in C^1(J; \mathbb{R}^+), \qquad q \notin L^1(J).$$

(H_4) For every $U > 0$ there exist two measurable functions σ, δ defined on J, such that for all $t \in J$, $|u| \le U$ and $p \in \mathbb{R}^N$

$$(Q(t,u,p), p) \le -\sigma(t)|p|^m, \tag{2.2}$$

$$|Q(t,u,p)| \le \delta(t)|p|^{m-1}. \tag{2.3}$$

Conditions (2.2) and (2.3) imply that $\delta \geq \sigma$ and $\delta \geq 0$. Note also that σ is allowed to take positive, negative and zero values.

The convexity and homogeneity of $G(u, \cdot)$ immediately give the following properties. For every $\lambda > 0$ and all $u, p \in \mathbb{R}^N$

$$G(u, \lambda p) = \lambda^m G(u, p), \qquad \nabla G(u, \lambda p) = \lambda^{m-1} \nabla G(u, p), \tag{2.4}$$

and for every $U > 0$ there exist positive constants c, c_1, c_2 such that

$$|\nabla G(u, p)| \leq c|p|^{m-1}, \tag{2.5}$$

$$c_1|p|^m \leq mG(u, p) = (\nabla G(u, p), p) \leq c_2|p|^m \tag{2.6}$$

for all $|u| \leq U$ and $p \in \mathbb{R}^N$. (From here on we write ∇ rather than ∇_p.)

From these properties we also obtain

$$G(u, 0) = 0, \qquad \nabla G(u, 0) = 0, \qquad \nabla_u G(u, 0) = 0, \tag{2.7}$$

while from (2.1) and (2.3) it follows that $f(0) = 0$, $Q(t, u, 0) = 0$. Hence $u = 0$ is a solution of (1.5).

Our proofs below are based on the well-known Liouville transformation, namely the change of independent variables $t \mapsto y$ given by

$$y = y(t) = \int_T^t q(s)ds, \qquad t \in J. \tag{2.8}$$

By (H$_3$) we have $y'(t) = q(t) > 0$ and $y(t) \nearrow \infty$ as $t \to \infty$, so that $J = [T, \infty)$ is diffeomorphically mapped into $\tilde{J} = [0, \infty)$. After a short calculation using the homogeneity of $G(u, \cdot)$, the system (1.5) is transformed into

$$(\nabla G(u, \dot{u}))^{\cdot} - \nabla_u G(u, \dot{u}) + f(u) = \tilde{Q}(y, u, \dot{u}), \qquad y \in \tilde{J}, \tag{2.9}$$

where $\dot{} = d/dy$ and, for $(y, u, v) \in \tilde{J} \times \mathbb{R}^N \times \mathbb{R}^N$,

$$\tilde{Q}(y, u, v) = \frac{1}{q} \left\{ \frac{Q(t, u, qv)}{q^{m-1}} - (m-1)\, \frac{q'}{q}\, \nabla G(u, v) \right\}. \tag{2.10}$$

The new system (2.9) has exactly the form (1.5), except that the term $q^m(t)f(u)$ is replaced simply by $f(u)$ and $Q(t, u, u')$ by $\tilde{Q}(y, u, \dot{u})$. For this system the hypotheses (H$_1$)-(H$_3$) still hold. The appropriate conditions on $\tilde{Q}$ replacing (2.2) and (2.3) are given by the following

Lemma 1. *For every $U > 0$ we have, for all $y \in \tilde{J}$, $|u| \le U$ and $v \in \mathbb{R}^N$,*

$$(\tilde{Q}(y,u,v),v) \le -\frac{S(t)}{q(t)}\,|v|^m, \qquad |\tilde{Q}(y,u,v)| \le \frac{D(t)}{q(t)}\,|v|^{m-1}, \tag{2.11}$$

with

$$S(t) = \begin{cases} \sigma(t) + c_1(m-1)\dfrac{q'(t)}{q(t)} & \text{if } q'(t) \ge 0 \\ \sigma(t) + c_2(m-1)\dfrac{q'(t)}{q(t)} & \text{if } q'(t) \le 0 \end{cases} \tag{2.12}$$

and

$$D(t) = \delta(t) + c(m-1)\frac{|q'(t)|}{q(t)}, \tag{2.13}$$

where σ, δ are the functions in (2.2) *and* (2.3)*, while c, c_1, c_2 are the constants in* (2.5) *and* (2.6)*. The quantities σ, δ, c, c_1 and c_2 may depend on U.*

Proof. We have by (2.10), (2.2) and (2.6)

$$\begin{aligned} (\tilde{Q}(y,u,v),v) &= \frac{1}{q}\left\{\frac{(Q(t,u,qv),qv)}{q^m} - (m-1)\frac{q'}{q}(\nabla G(u,v),v)\right\} \\ &\le \frac{1}{q}\left\{-\frac{\sigma(t)|qv|^m}{q^m} - (m-1)\frac{q'}{q}(\nabla G(u,v),v)\right\} \\ &\le \begin{cases} -\dfrac{|v|^m}{q}\left[\sigma(t) + c_1(m-1)\dfrac{q'}{q}\right] & \text{if } q'(t) \ge 0 \\ -\dfrac{|v|^m}{q}\left[\sigma(t) + c_2(m-1)\dfrac{q'}{q}\right] & \text{if } q'(t) \le 0, \end{cases} \end{aligned} \tag{2.14}$$

proving $(2.11)_1$. The inequality $(2.11)_2$ is obtained in the same way, using (2.3) and (2.5).

In the variational case, when

$$Q(t,u,p) = -\frac{g'(t)}{g(t)}\nabla G(u,p), \tag{2.15}$$

the above result can be improved. Setting $h = g'/g$, and

$$A = h + (m-1)q'/q,$$

we get

Lemma 2. *Let Q have the special form* (2.15)*. Then for every $U > 0$ we have*

$$(\tilde{Q}(y,u,v),v) \le -\tilde{\sigma}(y)|v|^m, \qquad |\tilde{Q}(y,u,v)| \le c\frac{|A(t)|}{q(t)}|v|^{m-1} \tag{2.16}$$

for all $y \in \tilde{J}$, $|u| \le U$ and $v \in \mathbb{R}^N$, where

$$\tilde{\sigma}(y) = \begin{cases} c_1 A(t)/q(t) & \text{if } A(t) \ge 0 \\ c_2 A(t)/q(t) & \text{if } A(t) < 0 \end{cases}$$

and c, c_1, c_2 are given by (2.5) *and* (2.6)*.*

Proof. From (2.10) and (2.15) we obtain

$$\begin{aligned} (\tilde{Q}(y,u,v),v) &= -\frac{1}{q}\left\{h\frac{(\nabla G(u,qv),qv)}{q^m} + (m-1)\frac{q'}{q}(\nabla G(u,v),v)\right\} \\ &= -\frac{1}{q}\left\{h + (m-1)\frac{q'}{q}\right\}(\nabla G(u,v),v) \end{aligned} \tag{2.17}$$

by $(2.4)_2$. Condition $(2.16)_1$ is now a consequence of (2.6). The estimate $(2.16)_2$ follows similarly using (2.5) and (2.10).

We conclude Section 2 with a useful boundedness theorem of independent interest.

Proposition. *Suppose $S(t) \ge 0$ in J, or $A(t) \ge 0$ for the variational case. Assume also that*

$$F(u) \to \infty \qquad \text{as } |u| \to \infty. \tag{2.18}$$

Then every solution of (1.5) *is bounded on J.*

Proof. Let $u = u(t)$ be a solution of (1.5) in J. Then $u(y)$ is a solution of (2.9) in $\tilde{J}$. By Theorem 8 of [12] we have the Hamilton identity

$$\{H(u,\dot{u}) + F(u)\}^{\cdot} = (\tilde{Q}(y,u,\dot{u}),\dot{u}),$$

where $H(u,v)$ is the Legendre transform of $G(u,v)$ in the variable v, namely

$$H(u,v) = (\nabla G(u,v),v) - G(u,v).$$

By $(2.11)_1$, or $(2.16)_1$, and by assumption, we get

$$\{H(u,\dot{u})+F(u)\}^{\cdot} \le 0 \qquad \text{in } \tilde{J},$$

so that $H(u,\dot{u})+F(u)$ is bounded above. Since $H(u,v)\ge 0$ by (H_1) it follows that $F(u)$ is bounded above in $\tilde{J}$. Hence (2.18) implies that $u(y)$ is bounded in $\tilde{J}$ which completes the proof.

A similar idea appears in [1, Theorem 4]. We have used this proposition in obtaining the final conclusions of Theorem B and C, as well as for Theorem A and the Matukuma equation (for the latter cases $F(u)=\int_0^u f(s)ds = u^2/2$ and $|u|^{p+1}/(p+1)$, respectively).

3. Main results

This section contains our main results and their proofs. Applications to special cases will be given in the next section.

We say that the forcing vector $Q=Q(t,u,p)$ in (1.5) is *tame* if for every $U>0$ there exists a constant $\gamma\ge 1$ such that

$$|Q(t,u,p)|\cdot|p| \le \gamma\,|(Q(t,u,p),p)|$$

for all $t\in J$, $|u|\le U$ and $p\in\mathbb{R}^N$. A similar definition applies for the forcing vector $\tilde{Q}=\tilde{Q}(y,u,v)$ in (2.9)-(2.10).

Since $|(Q,p)|/|Q|\cdot|p|$ is exactly the cosine of the angle between the vectors $Q=Q(t,u,p)$ and $-p$, the condition that Q be tame can be expressed geometrically by saying that the vectors Q and $-p$ are *bounded* away from orthogonality.

We note that when $N=1$ both Q and $\tilde{Q}$ are tame, with $\gamma=\tilde{\gamma}=1$. In Theorems 3 and 4 below we present other natural sufficient conditions guaranteeing that $\tilde{Q}$ is tame.

We can now state our first main result.

Theorem 1. *Let $\tilde{Q}$ be tame. Assume that for every $U>0$ there exists a continuous* *function $k = k(t)$ of bounded variation on J, such that*

$$S \geq kq \qquad \text{in } J, \tag{3.1}$$

$$kq \notin L^1(J), \tag{3.2}$$

$$\liminf_{t\to\infty} \int^t k^m(s)D(s)ds \Big/ \int^t k(s)q(s)ds < \infty, \tag{3.3}$$

where S and D are the functions defined in (2.12) *and* (2.13)*.*

Then every bounded solution of (1.5) *tends to zero as $t \to \infty$.*

Proof. We shall apply Theorem 4.1 of [11] to the transformed system (2.9). First, the basic assumptions (H_1)-(H_3) of Section 2 of [11] are satisfied, as direct consequences of (H_1), (H_2), $(2.11)_1$ and (3.1).

Next, condition $(C_1)'$ of [11] can be written, for (2.9), in the following form: for every $U > 0$ there exist two non-negative measurable functions $\tilde{\sigma}$, $\tilde{\delta} : \tilde{J} \to \mathbb{R}$ such that, for all $y \in \tilde{J}$, $|u| \leq U$ and $v \in \mathbb{R}^N$,

$$(\tilde{Q}(y,u,v), v) \leq -\tilde{\sigma}(y)|v|^m \qquad \text{and} \qquad |\tilde{Q}(y,u,v)| \leq \tilde{\delta}(y)|v|^{m-1}.$$

[Here $\mu = \nu = m - 1$.]

By (2.11) this condition is satisfied for (2.9) with

$$\tilde{\sigma}(y) = S(t)/q(t) \qquad \text{and} \qquad \tilde{\delta}(y) = D(t)/q(t), \tag{3.4}$$

where y and t and related by (2.8).

Condition (C_2) of [11] is exactly the assumption that $\tilde{Q}$ be tame.

The basic conditions (H_1)-(H_3), $(C_1)'$ and (C_2) of [11] are therefore satisfied for (2.9). It remains to check the specific hypotheses of Theorem 4.1, in Leoni's version in which (4.4) of [11] is replaced by (3.2) of [5].

For the system (2.9) these hypotheses take the form: *for every $U > 0$ there exists a* *continuous function $\tilde{k}$ of bounded variation on $\tilde{J}$ such that*

$$\tilde{k} \notin L^1(\tilde{J}); \tag{3.5}$$

$$0 \le \tilde{k}(y) \le \tilde{\sigma}(y), \qquad y \in \tilde{J}; \tag{3.6}$$

$$\liminf_{y\to\infty} \int^y \tilde{k}^m(s)\tilde{\delta}(s)ds \bigg/ \int^y \tilde{k}(s)ds < \infty. \tag{3.7}$$

Now we make the choice

$$\tilde{k}(y) = k(t),$$

with y and t related as usual by (2.8). Then (3.5) holds in view of (3.2) and the differential relation $dy = q(t)dt$. Condition (3.6) follows directly from (3.1) and $(3.4)_1$. Finally (3.7) is satisfied in view of $(3.4)_2$, (3.3) and, again, the relation $dy = q(t)dt$.

This being shown Theorem 4.1 implies that every bounded solution u of (2.9) satisfies

$$u(y) \to 0 \qquad \text{and} \qquad \dot{u}(y) \to 0 \qquad \text{as } y \to \infty.$$

Therefore, returning to the variable t, we see that

$$u(t) \to 0 \qquad \text{and} \qquad u'(t)/q(t) \to 0 \qquad \text{as } t \to \infty.$$

This completes the proof.

Remarks. 1. In general one cannot expect that $u'(t) \to 0$ as $t \to \infty$, at least when $q(t) \to \infty$ as $t \to \infty$. A particular example illustrating this fact was obtained by Wiman, who showed for equation (1.1) in the special case $q(t) = t^\beta$, $\beta > -1$, that both functions

$$t^{\beta/2}u(t) \qquad \text{and} \qquad t^{-\beta/2}u'(t)$$

tend to finite limits as $t \to \infty$. A more general conclusion of the same sort is given in [10], Theorem 7.

2. Condition (3.1) is actually required only for all sufficiently large t, because the conclusion of Theorem 1 refers to asymptotic behavior at ∞. A similar remark naturally applies in corresponding places throughout the paper (for instance, for conditions (i) in Theorems A-C of the introduction).

Corollary. *Let $\delta \in L^1(J)$ and assume $\tilde{Q}$ is tame. Suppose that for every $U > 0$ there exists a non-negative continuous function ψ of bounded variation on J, such that*

$$S \geq \psi \quad \text{in } J, \qquad \psi \notin L^1(J).$$

Then every bounded solution of (1.5) *tends to zero as $t \to \infty$.*

Proof. Put $d(t) = \exp\left(n \int^t \delta(s)ds\right)$, where $n = (2c+c_2)/cc_2(m-1)$, and choose $k = \psi/q$. Then

$$\begin{aligned} \frac{(dq)'}{dq} &= n\,\delta + \frac{q'}{q} \\ &= \begin{cases} \dfrac{2\delta}{c_2(m-1)} + \dfrac{1}{c(m-1)}\left[\delta + c(m-1)\dfrac{|q'|}{q}\right] & \text{if } q' \geq 0 \\ \dfrac{2}{c_2(m-1)}\left[\delta + c_2(m-1)\dfrac{q'}{q}\right] + \dfrac{1}{c(m-1)}\left[\delta + c(m-1)\dfrac{|q'|}{q}\right] & \text{if } q' < 0 \end{cases} \\ &\geq \frac{D}{c(m-1)}, \end{aligned}$$

since $\delta \geq \max\{0, \sigma\}$ and $S \geq \psi \geq 0$. Consequently in (3.3) we have

$$\int^t k^m D ds \leq c(m-1)\int^t k^m \frac{(dq)'}{dq} ds = c(m-1)\int^t (\psi d)^m \frac{(dq)'}{(dq)^{m+1}} ds.$$

Here $\psi\, d$ is bounded on J since $\delta \in L^1(J)$ and ψ is of bounded variation on J. Therefore

$$\int^t k^m D ds \leq \text{Const.}\left[-\frac{1}{(dq)^m}\right]^t \leq \text{Const.},$$

because $(dq)' \geq 0$ on J. Hence (3.1)-(3.3) are satisfied.

Finally, $k = \psi/q = \psi\, d(dq)^{-1}$, so that k, being the product of three BV functions, is itself of bounded variation. Consequently Theorem 1 applies and the proof is complete.

Our second main result concerns the variational case of (1.5), when Q is of the form (2.15), that is

$$Q(t, u, p) = -h(t)\nabla G(u, p), \qquad h \in C(J). \tag{3.8}$$

Here we define

$$A(t) = h(t) + (m-1)q'(t)/q(t), \qquad t \in J. \tag{3.9}$$

Theorem 2. *Suppose that there exists a non-negative continuous function k of bounded variation on J, such that*

$$A \geq kq \qquad \text{in } J, \tag{3.10}$$

$$kq \notin L^1(J), \tag{3.11}$$

$$\liminf_{t\to\infty} \int^t k^m(s)A(s)ds \Big/ \int^t k(s)q(s)ds < \infty. \tag{3.12}$$

Then every bounded solution of (1.5), (3.8) *tends to zero as $t \to \infty$.*

Proof. This is the same as for Theorem 1, with the exception that (2.11) is replaced by (2.16), and the tameness of $\tilde{Q}$ must be verified directly.

Noting that $A(t) \geq 0$ in J by (3.10), the relations (3.4) become for the present case

$$\tilde{\sigma}(y) = c_1 A(t)/q(t) \qquad \text{and} \qquad \tilde{\delta}(y) = cA(t)/q(t), \tag{3.13}$$

where y and t are related by (2.8), and c, c_1 are the constants in (2.5), (2.6). Therefore, choosing $\tilde{k}(y) = k(t)$ and using (3.13), the conditions (3.5)-(3.7) hold in view of (3.10)-(3.12).

It remains to see that $\tilde{Q}$ is tame. Fix $U > 0$. We have by (2.17), $(2.6)_1$ and (2.5)

$$\begin{aligned} |(\tilde{Q}(y,u,v),v)| &= \frac{A(t)}{q(t)}(\nabla G(u,v),v) \\ &\geq \frac{c_1}{c}\,\frac{A(t)}{q(t)}\,|\nabla G(u,v)|\cdot|v| = \frac{c_1}{c}\,|\tilde{Q}(y,u,v)|\cdot|v|, \end{aligned}$$

by (2.10), (3.8) and (3.9). Hence $\tilde{Q}$ is tame, with $\tilde{\gamma} = c/c_1$. This completes the proof of the theorem.

Theorem B of the introduction is the special case of Theorem 2 when $N = 1$, $m = 2$ and $G(u,p) = |p|^2/2$. Theorem C is the special case when $G(u,p) = |p|^2/2$ and $h(t) = 0$ in (3.8).

We have observed that $\tilde{Q}$ is tame when $N = 1$ and also in the variational case. The following results apply when these latter conditions are not satisfied.

Theorem 3. *Suppose that $q'(t) \geq 0$ in J and $(Q(t,u,p),p) \leq 0$ in $J \times \mathbb{R}^N \times \mathbb{R}^N$. Then if Q is tame, also $\tilde{Q}$ is tame.*

Proof. Fix $U > 0$. By (2.10) and the hypotheses of the theorem we have

$$\begin{aligned}|(\tilde{Q}(y,u,v),v)| &= \frac{1}{q}\left\{\frac{|(Q(t,u,qv),qv)|}{q^m} + (m-1)\frac{q'}{q}(\nabla G(u,v),v)\right\}\\ &\geq \frac{1}{q}\left\{\frac{1}{\gamma}\frac{|Q(t,u,qv)|\cdot|qv|}{q^m} + \frac{c_1}{c}(m-1)\,\frac{q'}{q}|\nabla G(u,v)|\cdot|v|\right\}\\ &= \frac{|v|}{q}\left\{\frac{1}{\gamma}\,\frac{|Q(t,u,qv)|}{q^{m-1}} + \frac{c_1}{c}(m-1)\,\frac{q'}{q}|\nabla G(u,v)|\right\}\end{aligned}$$

by (2.5) and (2.6). Now putting $\tilde{\gamma} = \max\{\gamma, c/c_1\}$ we obtain

$$\begin{aligned}|(\tilde{Q}(y,u,v),v)| &\geq \frac{1}{\tilde{\gamma}}\,\frac{|v|}{q}\left\{\frac{|Q(t,u,qv)|}{q^{m-1}} + (m-1)\frac{q'}{q}|\nabla G(u,v)|\right\}\\ &\geq \frac{1}{\tilde{\gamma}}|\tilde{Q}(y,u,v)|\cdot|v|\end{aligned} \tag{3.14}$$

by (2.10). Thus $\tilde{Q}$ is tame.

The next result will not be used in the later parts of the paper, but it is presented for its general interest.

Theorem 4. *Suppose for every $U > 0$ there exists numbers*

$$\theta_1 < c_1(m-1), \qquad \theta_2 > c_2(m-1) \tag{3.15}$$

such that the function

$$\tilde{S}(t) = \begin{cases}\sigma(t) + \theta_1\dfrac{q'(t)}{q(t)} & \text{if } q'(t) \geq 0\\[2ex] \sigma(t) + \theta_2\dfrac{q'(t)}{q(t)} & \text{if } q'(t) \leq 0\end{cases} \tag{3.16}$$

is non-negative in J; here σ is the function given in (2.2). *Then $\tilde{Q}$ is tame if Q is tame.*

Proof. Fix $U > 0$ and $y \in \tilde{J}$, $|u| \leq U$, $v \in \mathbb{R}^N$. Let $t \in J$ correspond to y through (2.8), and define

$$p = qv, \qquad \text{where } q = q(t).$$

We distinguish four cases.

1. $q'(t) \geq 0$, $(Q(t,u,p),p) \leq 0$. By the proof of Theorem 3, see (3.14), we get

$$|\tilde{Q}(y,u,v)| \cdot |v| \leq \gamma_1 |(\tilde{Q}(y,u,v),v)|,$$

where $\gamma_1 = \max\{\gamma, c/c_1\}$.

2. $q'(t) \geq 0$, $(Q(t,u,p),p) > 0$. Here by (2.10), (2.2), (2.6) and the fact that $\tilde{S}(t) \geq 0$ we have

$$\begin{aligned}(\tilde{Q}(y,u,v),v) &= \frac{1}{q}\left\{\frac{(Q(t,u,p),p)}{q^m} - (m-1)\frac{q'}{q}(\nabla G(u,v),v)\right\}\\ &\leq \frac{1}{q}\left\{-\sigma(t)|v|^m - c_1(m-1)\frac{q'}{q}|v|^m\right\} \qquad (3.17)\\ &\leq \frac{q'}{q^2}\{\theta_1 - c_1(m-1)\}|v|^m < 0,\end{aligned}$$

by $(3.15)_1$. Hence

$$|(\tilde{Q}(y,u,v))| \geq \frac{q'}{q^2}\{c_1(m-1) - \theta_1\}|v|^m.$$

On the other hand, because Q is tame we obtain from (2.5), (2.2) and the condition $(Q(t,u,p),p) > 0$

$$\begin{aligned}|\tilde{Q}(y,u,v)| \cdot |v| &\leq \frac{1}{q}\left\{\frac{|Q(t,u,p)|}{q^m} + (m-1)\frac{q'}{q}|\nabla G(u,v)|\right\}|v|\\ &\leq \frac{1}{q}\left\{\gamma\frac{(Q(t,u,p),p)}{q^m} + c(m-1)\frac{q'}{q}|v|^m\right\}\\ &\leq \frac{|v|^m}{q}\left\{-\gamma\sigma(t) + c(m-1)\frac{q'}{q}\right\}\\ &\leq \frac{q'}{q^2}\{\gamma\theta_1 + c(m-1)\}|v|^m,\end{aligned} \qquad (3.18)$$

since $\tilde{S}(t) \geq 0$. Choose

$$\gamma_2 = \frac{\gamma\theta_1 + c(m-1)}{c_1(m-1) - \theta_1}, \qquad (3.19)$$

so that $\gamma_2 > c/c_1$. Then by (3.17) and (3.18) we obtain

$$|\tilde{Q}(y,u,v)| \cdot |v| \leq \gamma_2 |(\tilde{Q}(y,u,v),v)|. \qquad (3.20)$$

3. $q'(t) < 0$, $(Q(t,u,p),p) < 0$. By (2.10), (2.2), (2.6), and since $\tilde{S}(t) \geq 0$, we have

$$(\tilde{Q}(y,u,v),v) \leq \frac{1}{q}\left\{-\sigma(t)|v|^m + (m-1)\frac{|q'|}{q}c_2|v|^m\right\}$$
$$\leq \frac{|q'|}{q^2}\{-\theta_2 + c_2(m-1)\} < 0$$

by $(3.15)_2$. Hence

$$|(\tilde{Q}(y,u,v),v)| \geq -\frac{1}{q}\left\{\frac{(Q(t,u,p),p)}{q^m} + c_2(m-1)\frac{|q'|}{q}|v|^m\right\}. \tag{3.21}$$

On the other hand since Q is tame we find from (2.5) and the condition $(Q(t,u,p),p) < 0$

$$\begin{aligned}|\tilde{Q}(y,u,v)|\cdot|v| &\leq \frac{|v|}{q}\left\{\frac{|Q(t,u,p)|}{q^m} + (m-1)\frac{|q'|}{q}\cdot|\nabla G(u,v)|\right\}\\ &\leq \frac{1}{q}\left\{-\frac{\gamma}{q^m}(Q(t,u,p),p) + c(m-1)\frac{|q'|}{q}|v|^m\right\}.\end{aligned} \tag{3.22}$$

Choose now

$$\gamma_3 = \frac{\theta_2\gamma + c(m-1)}{\theta_2 - c_2(m-1)},$$

so that $\gamma_3 > \gamma$ and

$$(\gamma_3 - \gamma)\theta_2 = (c_2\gamma_3 + c)(m-1). \tag{3.23}$$

We assert that

$$|\tilde{Q}(y,u,v)|\cdot|v| \leq \gamma_3|(\tilde{Q}(y,u,v),v)|. \tag{3.24}$$

Indeed by (3.21) and (3.22) it is enough to show that

$$-\gamma\frac{(Q(t,u,p),p)}{q^m} + c(m-1)\frac{|q'|}{q}|v|^m \leq -\gamma_3\left\{\frac{(Q(t,u,p),p)}{q^m} + c_2(m-1)\frac{|q'|}{q}|v|^m\right\},$$

namely

$$-(\gamma_3-\gamma)\frac{(Q(t,u,p),p)}{q^m} \geq (\gamma_3 c_2 + c)(m-1)\frac{|q'|}{q}. \tag{3.25}$$

However, by (2.2) and the fact that $\tilde{S}(t) \geq 0$ we have

$$-(Q(t,u,p),p) \geq \sigma(t)q^m|v|^m \geq \theta_2 q^m|v|^m\frac{|q'|}{q};$$

thus (3.25) follows from (3.23). This completes the proof of (3.24).

4. $q'(t) < 0$, $(Q(t,u,p),p) \geq 0$. This case cannot occur. Indeed, since $\tilde{S}(t) \geq 0$ by assumption, there results $\sigma(t) \geq \theta_2 |q'(t)|/q(t) > 0$ and consequently $(Q(t,u,p),p) < 0$ by (2.2).

The assertion of the theorem is now proved by choosing $\tilde{\gamma} = \max\{\gamma_2, \gamma_3\}$.

Remark. The condition $\tilde{S}(t) \geq 0$ in J should be compared with the related hypothesis of Theorem 1,

$$S(t) \geq k(t)q(t) > 0 \qquad \text{in } J.$$

In fact the following result holds.

Corollary. *Suppose that*

$$S(t) \geq \text{Const.}\, |q'(t)|/q(t) \qquad \textit{for all } t \in J. \tag{3.26}$$

Then $\tilde{Q}$ is tame if Q is tame.

Proof. By (3.16) and (2.12) we have

$$\begin{aligned} \tilde{S}(t) &= S(t) - \frac{|q'(t)|}{q(t)} \begin{cases} c_1(m-1) - \theta_1 & \text{if } q'(t) \geq 0 \\ \theta_2 - c_2(m-1) & \text{if } q'(t) < 0 \end{cases} \\ &\geq S(t) \begin{cases} 1 - \kappa[c_1(m-1) - \theta_1] & \text{if } q'(t) \geq 0 \\ 1 - \kappa[\theta_2 - c_2(m-1)] & \text{if } q'(t) < 0, \end{cases} \end{aligned}$$

where $1/\kappa$ is the constant in (3.26). Now it is enough to choose

$$\theta_1 \in \big(c_1(m-1) - 1/\kappa,\, c_1(m-1)\big), \qquad \theta_2 \in \big(c_2(m-1),\, c_2(m-1) + 1/\kappa\big).$$

Then (3.15) is satisfied and $\tilde{S}(t) > 0$ in J. Hence Theorem 4 can be applied and the proof is complete.

4. Applications and extensions

1. Holonomic mechanical systems

A well-known example studied by Salvadori [13] is the holonomic bilaterally constrained mechanical system

$$(\nabla G(u,u'))' - \nabla_u G(u,u') + q^2(t)f(u) = -L(t,u)u', \qquad t \in J, \tag{4.1}$$

where

$$G(u,p) = \tfrac{1}{2}(M(u)p,p).$$

Here L is a continuous, bounded and uniformly positive definite $N \times N$ matrix; M is a positive definite $N \times N$ matrix of class C^1; and q is a positive non-decreasing function of class C^1; see [13, Sections 3.2 and 4.5]. The function f is assumed to satisfy hypothesis (H_2) in Section 2.

Under these conditions Salvadori showed that, *if also*

$$q'/q^2 \text{ is bounded in } J, \tag{4.2}$$

then any bounded solution of (4.1) *approaches zero as* $t \to \infty$.

[Salvadori actually considered only uniform asymptotic stability, rather than asymptotic stability of bounded solutions. For simplicity we have restated his theorem in the context of our earlier results].

Here we present a generalization of Salvadori's theorem, in which condition (4.2) is *not* required.

Criterion of Salvadori-type. *Let L, M, f and q verify the conditions stated above, except* (4.2). *Then every bounded solution of* (4.1) *tends to zero as* $t \to \infty$.

This criterion is a consequence of the following theorem for the system (1.5) under the natural hypotheses (H_1)-(H_4); in particular the condition $q'(t) \geq 0$ in J is not assumed.

Theorem 5. *Let $\tilde{Q}$ be tame. Assume that for every $U > 0$ there exists a non-negative* C^1 *function ψ such that*

$$\psi \notin L^1(J), \qquad \psi/q \in BV(J), \tag{4.3}$$

$$S \geq \psi \quad \text{in } J, \tag{4.4}$$

$$\liminf_{t\to\infty} \int^t \left[\frac{\psi(s)}{q(s)}\right]^m \delta(s)ds \Big/ \int^t \psi(s)ds < \infty, \tag{4.5}$$

and either

$$q' \leq \text{Const.}\, q^{m+1}/\psi^{m-1} \quad \text{or} \quad \psi' \leq \text{Const.}\, q^m/\psi^{m-2} \quad \text{in } J, \tag{4.6}$$

where S is defined by (2.12)*, and σ, δ are the functions given in* (H_4)*.*

Then every bounded solution of (1.5) *tends to zero as $t \to \infty$.*

Proof. Take $k = \psi/q$ in Theorem 1, so that $k \in CBV(J)$. Then (3.1) follows from (4.4), and (3.2) from $(4.3)_1$.

As shown in the proof of the corollary of Theorem 1 and using the same notation,

$$D \leq c(m-1)\left[n\delta + q'(t)/q(t)\right].$$

Consequently

$$\int^t k^m(s)D(s)ds \leq c(m-1)\left\{n\int^t \left[\frac{\psi(s)}{q(s)}\right]^m \delta(s)ds + \int^t \left[\frac{\psi(s)}{q(s)}\right]^m \frac{q'(s)}{q(s)}\,ds\right\}.$$

If $(4.6)_1$ holds, then clearly

$$\int^t k^m(s)D(s)ds \leq \text{Const.}\left\{\int^t \left[\frac{\psi(s)}{q(s)}\right]^m \delta(s)ds + \int^t \psi(s)ds\right\}$$

and (3.3) therefore follows from (4.5). If $(4.6)_2$ holds, then by integration by parts

$$\int^t \psi^m(s)\frac{q'(s)}{q^{m+1}(s)}\,ds = -\frac{1}{m}\left[\frac{\psi(s)}{q(s)}\right]^m\Bigg|^t + \int^t \left[\frac{\psi(s)}{q(s)}\right]^m \frac{\psi'(s)}{\psi(s)}\,ds$$

$$\leq \text{Const.}\left(1 + \int^t \psi(s)ds\right).$$

Combining the two previous inequalities now yields

$$\int^t k^m(s)D(s)ds \le \text{Const.}\left(\int^t \left[\frac{\psi(s)}{q(s)}\right]^m \delta(s)ds + 1 + \int^t \psi(s)ds\right),$$

so that (3.3) is again satisfied, by (4.5) and $(4.3)_1$.

Thus the conditions of Theorem 1 hold, and all bounded solutions of (1.5) tend to zero as $t \to \infty$. This completes the proof.

Theorem 6. *If* $1/q$ *is of bounded variation on* J*, then Theorem 5 remains valid when* $(4.3)_2$ *and* (4.6) *are replaced by* $\psi \in BV(J)$*.*

Proof. Take $k = \psi/q$ as in the proof of Theorem 5. Now we have

$$\int^t \left[\frac{\psi(s)}{q(s)}\right]^m \frac{q'(s)}{q(s)}ds \le \text{Const.}\int^t \frac{|q'(s)|}{q^2(s)}ds,$$

since ψ and $1/q$ are bounded. The last integral, however, is convergent since $1/q$ is of bounded variation on J. It now follows, as in the proof of Theorem 5, that

$$\int^t k^m(s)D(s)ds \le \text{Const.}\left(\int^t \left[\frac{\psi(s)}{q(s)}\right]^m \delta(s)ds + 1\right)$$

and (3.3) then follows from (4.5) and $(4.3)_1$. Finally, (3.1) and (3.2) are just (4.4) and $(4.3)_1$, while k is of bounded variation since both ψ and $1/q$ are. This completes the proof.

Corollary A2 is the special case of Theorem 6 when $\sigma = \delta = 0$, since then (4.4) implies $q' \ge 0$ in J, so that $1/q$ is of bounded variation.

Corollary. *Let the hypotheses of Theorem 5 or Theorem 6 be satisfied, except that* (4.5) *is replaced by*

$$\delta \le \text{Const.}\, q^m/\psi^{m-1} \qquad \text{on } J. \tag{4.7}$$

Then every bounded solution of (1.5) *tends to zero as* $t \to \infty$*.*

Proof. The pointwise condition (4.7) obviously implies (4.5). Consequently Theorem 5 or Theorem 6 can be applied.

We now prove the Salvadori criterion stated above. First, since $Q(t,u,p) = -L(t,u)p$ and L is bounded and uniformly positive definite, we get

$$|(Q(t,u,p),p)| = (L(t,u)p,p) \geq b\,|p|^2 \geq \frac{b}{\ell}\,|Q(t,u,p)|\cdot|p|,$$

where $b = \text{Const.} > 0$ and $\ell = \sup\{\|L(t,u)\| : t \in J,\ u \in \mathbb{R}^N\}$. Thus Q is tame with $\gamma = \ell/b$.

We can now apply Theorem 3, since $q'(t) \geq 0$ by assumption and $(Q(t,u,p),p) = -(L(t,u)p,p) \leq 0$. Hence $\tilde{Q}$ is tame in (4.1).

In view of the regularity and positive definiteness of M, it is clear that condition (H_1) holds for (4.1) with $m = 2$, while conditions (H_2) and (H_3) have already been directly assumed. Finally (H_4) is verified with $\sigma(t) = \text{Const.} = b$ and $\delta(t) = \text{Const.} = \ell$.

Take $\psi(t) = 1/t$ in the corollary above. Then $(4.3)_{1,2}$ are satisfied since $\psi(t)/q(t) = 1/tq(t)$ is decreasing ($q'(t) \geq 0$ in J). Similarly $S \geq \sigma$, so (4.4) and $(4.6)_2$ hold ($\psi' < 0$). Finally, condition (4.7) is obvious since $q(t) \geq \text{Const.} > 0$. Consequently the previous corollary applies and the proof is complete.

When $N = 1$, or more generally whenever Q is tame in (4.1) - as for example when the matrix L is a multiple of the identity - it is possible to weaken significantly the assumptions on L in the Salvadori criterion. In particular it is enough to require merely that

$$(L(t,u)p,p) \geq \frac{b}{t}, \qquad \|L(t,u)\| \leq \ell\, t^{m-1} q^m(t)$$

for $t \in J$, $|u| \leq U$ and $|p| = 1$, where b and ℓ are positive constants. Indeed in this case we can take $\sigma(t) = b/t$ and $\delta(t) = \ell\, t^{m-1} q^m(t)$ in (H_4), so that the corollary above again applies with $\psi(t) = 1/t$.

2. Radial Matukuma-type equations

We consider here a generalized version of (1.4),

$$u'' + \frac{n-1}{t}\,u' + q^2(t) f(u) = 0, \qquad t > 0, \tag{4.8}$$

where $N = 1$, $n > 1$, and f, q are assumed to satisfy conditions (H_2)-(H_3) with $J = [T,\infty)$, $T > 0$.

Theorem 7. *Suppose that there exists a non-negative* C^1 *function* ψ *such that*

$$\psi \notin L^1(J), \qquad t\psi \in L^\infty(J), \qquad \psi/q \in BV(J), \tag{4.9}$$

$$\psi'(t) \le \text{Const.}\, q^2(t); \qquad tq'(t) + \theta q(t) \ge 0, \quad \theta \in (0, n-1), \tag{4.10}$$

$$\liminf_{t\to\infty} \int^t \left[\frac{\psi(s)}{q(s)}\right]^2 \frac{ds}{s} \Big/ \int^t \psi(s)ds < \infty. \tag{4.11}$$

Then every (ultimately) bounded solution of (4.8) *tends to zero as* $t \to \infty$.

Proof. We apply Theorem 5 with $N = 1$, $G(p) = |p|^2/2$, $m = 2$, and $\sigma(t) = \delta(t) = (n-1)/t$. Then (2.6) holds with $c_1 = c_2 = 1$, and (2.12) becomes

$$S(t) = \frac{n-1}{t} + \frac{q'(t)}{q(t)}, \qquad t \in J.$$

Consequently (4.4) follows from $(4.10)_2$ and $(4.9)_2$. Also $(4.6)_2$ is satisfied in view of $(4.10)_1$. Finally $(4.3)_{1,2}$ hold and (4.5) is a consequence of (4.11). This completes the proof.

Remarks. The same result applies to the m-Laplace equation

$$(|u'|^{m-2}u')' + \frac{n-1}{t}|u'|^{m-2}u' + q^m(t)f(u) = 0, \qquad m > 1, \tag{4.12}$$

provided $(4.10)_1$ is replaced by $\psi' \le \text{Const.}\, q^m/\psi^{m-2}$, the θ condition in $(4.10)_2$ is replaced by $\theta \in (0,\, (n-1)/(m-1))$, and the exponent 2 in (4.11) is replaced by m.

The condition $\theta \in (0,\, n-1)$ in $(4.10)_2$ cannot be replaced by $\theta \in (0,\, n-1]$. Indeed, if $n = 2$, $q(t) = 1/t$, then $(4.10)_2$ holds with $\theta = 1 = n-1$, but the transformed equation (2.9) takes the form $\ddot{u} + f(u) = 0$, and has periodic solutions.

Examples. 1. $\psi(t) = \text{Const.}/t$. Then $(4.9)_{1,2}$ and $(4.10)_1$ hold automatically, while condition (4.11) becomes

$$\liminf_{t\to\infty} \frac{1}{\log t} \int^t \left[\frac{\psi(s)}{q(s)}\right]^2 \frac{ds}{s} < \infty.$$

Since ψ/q is assumed to be of bounded variation on J, see $(4.9)_3$, it follows that ψ/q is bounded; thus (4.11) is also automatically satisfied.

The specific Matukuma function $q(t) = 1/\sqrt{1+t^2}$ is covered by this example, with $\theta = 1$ in $(4.10)_2$.

2. The case $\psi(t) = \text{Const.}/t^\alpha$, $\alpha > 1$, fails since $\psi \in L^1(J)$.

3. $\psi(t) = \text{Const.}/t \log t$. Again $(4.9)_{1,2}$ and $(4.10)_1$ hold, while (4.11) becomes

$$\liminf_{t\to\infty} \frac{1}{\log\log t} \int^t \left[\frac{\psi(s)}{q(s)}\right]^2 \frac{ds}{s} < \infty.$$

To investigate this condition, we make the principal assumption

$$t\,q(t) \geq \text{Const.}/\log^\alpha t, \qquad \alpha \in (0,1], \tag{4.13}$$

(when $\alpha > 1$, condition $(4.9)_3$ fails). Then

$$\int^t \left[\frac{\psi(s)}{q(s)}\right]^2 \frac{ds}{s} \leq \text{Const.} \int^t (\log s)^{2(\alpha-1)} ds = \begin{cases} -(\log t)^{2\alpha-1} + \text{Const.} & \text{if } \alpha < 1/2 \\ \log\log t + \text{Const.} & \text{if } \alpha = 1/2 \\ (\log t)^{2\alpha-1} + \text{Const.} & \text{if } \alpha > 1/2. \end{cases}$$

Hence (4.11) is satisfied if and only if $\alpha \leq 1/2$.

4. $\psi(t) = \text{Const.}\, q(t)/(\log t)^{1-\alpha}$, with q obeying (4.13). Here $(4.9)_1$ and $(4.9)_3$ are satisfied, while $(4.9)_2$, $(4.10)_1$ respectively become

$$tq(t) \leq \text{Const.}\,(\log t)^{1-\alpha} \quad \text{and} \quad q'(t) \leq \text{Const.}\, q^2(t)(\log t)^{1-\alpha}. \tag{4.14}$$

Finally from (4.13)

$$\int^t \left[\frac{\psi(s)}{q(s)}\right]^2 \frac{ds}{s} \Big/ \int^t \psi(s)ds \leq \frac{\text{Const.}}{\log\log t} \int^t \frac{ds}{s(\log s)^{2(1-\alpha)}}.$$

Therefore condition (4.11) again holds if and only if $\alpha \leq 1/2$.

The function $q^2(t) = 1/(1+t^2)\log(1+t^2)$ noted in the introduction is covered by both cases 3 and 4, with $\theta \in (1, n-1)$, $n > 2$, in $(4.10)_2$.

Comparing examples 3 and 4, we see that under the assumption (4.13) *both* $(4.10)_2$ *and* $\alpha \le 1/2$ *are needed.* On the other hand, in example 3 one also requires $(4.9)_3$, that is

$$1/tq(t)\log t \in BV(J), \tag{4.15}$$

while example 4 needs additionally (4.14) instead of (4.15). The conditions (4.14) and (4.15) are independent, so cases 3 and 4 provide different sufficient conditions for asymptotic stability.

The case $\alpha = 1/2$ of (4.13), namely the condition

$$q(t) \ge \frac{\text{Const.}}{t\sqrt{\log t}} \qquad \text{in } J,$$

effectively defines the borderline case for the validity of Theorem 7, as shown by the following

Theorem 8. *Suppose* $n > 2$ *and that*

$$\int^{\infty} sq^2(s)ds < \infty. \tag{4.16}$$

Then there exists a (bounded) solution u *of* (4.8) *on* $J = [T,\infty) \subset (0,\infty)$ *such that*

$$u(t) \to \text{limit} > 0 \qquad \textit{as } t \to \infty. \tag{4.17}$$

The set of attainable limits in (4.17) *is dense in* $\mathbb{R}$.

Proof. Consider the initial value problem for equation (4.8)

$$u(T_1) = a, \qquad u'(T_1) = 0, \tag{4.18}$$

where $a \in \mathbb{R}$ and where $T_1 \ge T > 0$ will be chosen later. By standard theory this problem has a local solution, and by Theorem 3 of [10] it can be continued to all of J, as required. The hypotheses of [10] are satisfied with $g(t) = t^{n-1}$ and $F(t,u)$ replaced by $q^2(t)F(u)$; moreover the four conditions (i)-(ii)$'$ on page 106 of [10] clearly hold since $F(u) \ge 0$ and $q'/q \in L^1_{loc}(J)$.

We now show that this solution has a limit as $t \to \infty$. First, integration of (4.8) then gives

$$u'(t) = -\frac{1}{t^{n-1}} \int_{T_1}^{t} s^{n-1} q^2(s) f(u(s)) ds, \qquad t \in [T_1, \infty), \tag{4.19}$$

and in turn

$$\begin{aligned} u(t) &= a - \int_{T_1}^{t} \frac{1}{r^{n-1}} \int_{T_1}^{r} s^{n-1} q^2(s) f(u(s)) ds\, dr \\ &= a - \int_{T_1}^{t} s^{n-1} q^2(s) f(u(s)) \int_{s}^{t} \frac{dr}{r^{n-1}}\, ds \\ &= a - \frac{1}{n-2} \int_{T_1}^{t} s^{n-1} q^2(s) f(u(s)) [s^{2-n} - t^{2-n}] ds \end{aligned} \tag{4.20}$$

by Fubini's theorem.

Assume $a > 0$ without loss of generality, and fix $\varepsilon \in (0, a/2)$. Denote by $B > 0$ the maximum value of $f(u)$ for $u \in [a - \varepsilon, a]$. Now choose $T_1 = T_1(\varepsilon) \geq T$, so that

$$\frac{B}{n-2} \int_{T_1}^{\infty} s q^2(s) ds \leq \varepsilon. \tag{4.21}$$

This can be done by virtue of (4.16).

We assert that

$$u(t) \geq a - \varepsilon \qquad \text{for } t \geq T_1. \tag{4.22}$$

To see this, let $[T_1, T_2)$, $T_2 \leq \infty$, be the maximal half-open interval for which (4.22) holds. Clearly $T_2 > T_1$ by (4.18) and (4.19). If $T_2 < \infty$ then by (4.20) and (4.21)

$$u(T_2) > a - \frac{B}{n-2} \int_{T_1}^{T_2} s q^2(s) ds \geq a - \varepsilon.$$

This contradicts the maximality of T_2. Hence $T_2 = \infty$ and (4.22) is proved.

From (4.22), (4.19) and the fact that $a - \varepsilon > 0$, we get $u'(t) < 0$ for $t > T_1$. Therefore $u(t) \searrow$ limit $\in [a - \varepsilon, a)$ as $t \to \infty$. Since $\varepsilon > 0$ and $a > 0$ are arbitrary, we have thus obtained the existence of a family of solutions on J of (4.8) having a dense set of limits as $t \to \infty$. This completes the proof.

Corollary. *Under the hypotheses of Theorem 8 the rest state $u = 0$ is not an attractor for bounded solutions of* (4.8).

Remarks. 1. It is evident that conditions (4.9)-(4.11) are incompatible with (4.16), since the conclusions of Theorems 6 and 7 cannot hold simultaneously. The incompatibility of $(4.9)_1$, (4.11) and (4.16) also follows directly from the inequality

$$\left[\int^t \left[\frac{\psi(s)}{q(s)}\right]^2 \frac{ds}{s} \Big/ \int^t \psi(s)ds\right] \cdot \left[\int^t sq^2(s)ds\right] \geq \int^t \psi(s)ds, \qquad t \in J. \tag{4.23}$$

[To prove (4.23), we have

$$\int^t \psi(s)ds \leq \left[\int^t \left[\frac{\psi(s)}{q(s)}\right]^2 \frac{ds}{s}\right]^{1/2} \cdot \left[\int^t sq^2(s)ds\right]^{1/2}$$

by Hőlder's inequality.]

2. If the function q is assumed to be locally Hőlder continuous on $[0,\infty)$, as is the case, for example, for the Matukuma equation, then one can obtain a family of positive regular solutions of (4.8), which are defined in the entire interval $[0,\infty)$ and tend to non-zero limits as $t \to \infty$, provided that (4.17) holds and the function $f = f(u)$ is monotone in $[0,\infty)$, see [8] and [15]. The set of limits ℓ which are attainable in this way include all sufficiently small $\ell > 0$ when f is non-decreasing, or all sufficiently large ℓ when f is non-increasing.

3. A canonical example

Consider the equation

$$u'' + h(t)u' + q^2(t)f(u) = 0, \qquad t \in J, \tag{4.24}$$

where $J = [T,\infty)$, $T > 0$, and where

$$th(t) \geq b\,t^\alpha, \qquad b = \text{constant} > 0, \quad \alpha = \text{constant} \geq 0, \tag{4.25}$$

$$tq(t) \quad \text{is non-decreasing in } J. \tag{4.26}$$

For the linear case of (4.24), see [1, page 326], and [4] with $th(t) = b\,t^{\alpha}$, $tq(t) = \sqrt{c}\,t^{\beta}$, where b, c are positive constants.

Here we apply the corollary of Theorems 5 and 6, with $\psi(t) = \text{Const.}/t$. Then (4.3) holds by (4.26). Next by (4.25) we can take $\sigma(t) = \text{Const.}\,t^{\alpha-1}$ in (2.2), so that

$$S(t) = \sigma(t) + \frac{q'(t)}{q(t)} = b\,t^{\alpha-1} + \frac{[tq(t)]'}{tq(t)} - \frac{1}{t} \geq b\,t^{\alpha-1} - \frac{1}{t}$$

by (2.12), (4.26), and the fact that $m = 2$, $c_1 = c_2 = 1$. Thus (4.4) is satisfied provided $\alpha > 0$. Moreover $(4.6)_2$ is obviously valid.

It remains to check condition (4.7), namely, since we can take $\delta = h$ in (2.3),

$$h(t) \leq \text{Const.}\,tq^2(t) \qquad \text{in } J. \tag{4.27}$$

Combining (4.25) and (4.27) then gives the condition

$$b\,t^{\alpha} \leq t\,h(t) \leq \text{Const.}\,[tq(t)]^2. \tag{4.28}$$

It follows now that *if* (4.26), (4.28) *hold and* $\alpha > 0$, *then the corollary of Theorems 5 and 6 applies and every bounded solution of* (4.24) *tends to zero as* $t \to \infty$.

Note that (4.28) implies in particular

$$tq(t) \geq \text{Const.}\,t^{\alpha/2}. \tag{4.29}$$

For the case treated in [4], namely $q(t) = \sqrt{c}\,t^{\beta-1}$, this gives specifically

$$\alpha \leq 2\beta, \tag{4.30}$$

a condition already found for asymptotic stability in [4].

If (4.26) is strengthened to

$$q \quad \text{non-decreasing in } J,$$

then it is clear that

$$S(t) \geq \text{Const.}\,t^{\alpha-1},$$

and so (4.4) then holds also when $\alpha = 0$.

The case $q(t) = \sqrt{c}\, t^{\beta-1}$, $\beta > 0$, is of particular interest. Here (4.29) reduces exactly to (4.30), and (4.26) certainly holds. Hence *every bounded solution of* (4.24) *tends to zero as* $t \to \infty$, *provided* $0 < \alpha \le 2\beta$ *and* $b\, t^\alpha \le t\, h(t) \le \text{Const.}\, t^{2\beta}$.

Corollary 8 of [1] applies to this case *whenever additionally* $\alpha \ge \beta$, that is, in view of (4.30), whenever $\beta \le \alpha \le 2\beta$, $\alpha > 0$, and (4.28) holds. On the other hand, when $\alpha < \beta$ Corollary 8 of [1] is not generally applicable, and indeed fails whenever $\liminf_{t\to\infty} t^{1-\alpha} h(t) < \infty$.

A second special case of interest occurs when the damping function h in (4.24) is bounded, and bounded from zero. In this case we can take $\alpha = 1$ in (4.25), so that (4.26) and (4.28) reduce to the simple conditions

$$tq(t) \quad \text{non-decreasing in } J, \qquad tq^2(t) \ge \text{Const.}$$

4. More general systems

Extensions to the system

$$(\nabla G(u, u'))' - \nabla_u G(u, u') + \nabla_u \Phi(t, u) = Q(t, u, u'), \qquad t \in J, \tag{4.31}$$

are also possible, provided the function $\Phi(t, u)$ has asymptotic behavior corresponding to $q^m(t)F(u)$, as in the case of the system (1.5).

Making the change of variables (2.8) the system (4.31) becomes

$$(\nabla G(u, \dot u))^{\cdot} - \nabla_u G(u, \dot u) + \frac{\nabla_u \Phi(t, u)}{q^m(t)} = \tilde Q(y, u, \dot u), \qquad y \in \tilde J, \tag{4.32}$$

where $\tilde Q$ is given by (2.10). In order to apply Theorem 4.1 of [11] to the system (4.32), as in the proof of Theorem 1 above, we must verify, for the function

$$\tilde F(y, u) = \frac{\Phi(t, u)}{q^m(t)},$$

the hypothesis (H_2) of [11]. It is easy to see that this requires exactly the following condition:

($\tilde{H}_2$) $\Phi \in C^1(J \times \mathbb{R}^N; \mathbb{R})$ and $\Phi(t, 0) = 0$. For all u_0, U with $0 < u_0 \le U$ there exist a constant $\kappa > 0$ and a non-negative function $\varphi \in L^1(J)$ such that

$$(\nabla_u \Phi(t,u), u) \ge \kappa\, q^m(t) \qquad \text{whenever } t \in J \text{ and } |u| \in [u_0, U],$$

$$\Phi_t(t,u) - m\frac{q'(t)}{q(t)}\Phi(t,u) \le q^m(t)\varphi(t) \qquad \text{whenever } t \in J \text{ and } |u| \le U.$$

The assumption ($\tilde{H}_2$) being made, all the previous conclusions carry over without change to the system (4.31).

Acknowledgement. The first author is a member of *Gruppo Nazionale di Analisi Funzionale e sue Applicazioni* of the *Consiglio Nazionale delle Ricerche.* This research has been partly supported by the Italian *Ministero della Università e della Ricerca Scientifica e Tecnologica.*

References

[1] Ballieu, R. J. & Peiffer, K., Attractivity of the origin for the equation $\ddot{x} + f(t,x,\dot{x})|\dot{x}|^\alpha \dot{x} + g(x) = 0$, *J. Math. Anal. Appl.* **65** (1978), 321–332.

[2] Cesari, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, 3rd ed., Springer–Verlag, Berlin, 1970.

[3] Corne, J. L., On the asymptotic stability, *Ann. Soc. Sci. Bruxelles Sér. I* **87** (1973), 217–235.

[4] Harris, W. A., Pucci, P. & Serrin, J., Asymptotic behavior of solutions of a nonstandard second order differential equation, *Diff. and Integral Equations* **6** (1993), 1201–1215.

[5] Leoni, G., A note on a theorem of Pucci and Serrin, to appear in *J. Diff. Equations* **113** (1994).

[6] Li, Y., On the positive solutions of the Matukuma equation, *Duke Math. J.* **70** (1993), 575–589.

[7] Li, Y. & Ni, W.-M., On conformal scalar curvature equations in $\mathbb{R}^n$, *Duke Math. J.* **57** (1988), 895–924.

[8] Naito, M., A note on bounded positive entire solutions of semilinear elliptic equations, *Hiroshima Math. J.* **14** (1984), 211–214.

[9] Prodi, G., Un'osservazione sugl'integrali dell'equazione $\ddot{y} + A(x)y = 0$ nel caso $A(x) \to +\infty$ as $x \to \infty$, *Rend. Accad. Naz. Lincei* **8** (1950), 462–464.

[10] Pucci, P. & Serrin, J., Continuation and limit properties for solutions of strongly nonlinear second order differential equations, *Asymptotic Anal.* **4** (1991), 97–160.

[11] —, Precise damping conditions for global asymptotic stability for nonlinear second order systems, *Acta Math.* **170** (1993), 275–307.

[12] —, On the derivation of Hamilton's equations, *Archive Rational Mech. Anal.* **125** (1994), 297–310.
[13] Salvadori, L., Famiglie ad un parametro di funzioni di Liapunov nello studio della stabilità, *Ist. Naz. di Alta Matematica, Symposia Matematica* **4** (1971), 307–330.
[14] Sansone, G., *Equazioni differenziali nel campo reale*, Vol II, 2nd edition, Zanichelli Editore, Bologna, Italy, 1949.
[15] Usami, H., On bounded positive entire solutions of semilinear elliptic equations, *Funkc. Ekv.* **29** (1986), 189–195.
[16] Wiman, A., Űber eine Stabilitätsfrage in the Theorie der linearen Differentialgleichungen, *Acta Math.* **66** (1936), 121–145.

PATRIZIA PUCCI
Dipartimento di Matematica
Università di Perugia
Via Vanvitelli, 1
I–06100 Perugia
Italy

JAMES SERRIN
Department of Mathematics
University of Minnesota
Minneapolis, MN 55455
U.S.A.