A Strong Maximum Principle and a Compact Support Principle for Singular Elliptic Inequalities

Patrizia Pucci, James Serrin & Henghui Zou

Abstract. Vazquez in 1984 established a strong maximum principle for the classical m–Laplace differential inequality

$$\Delta_m u - f(u) \le 0,$$

where $\Delta_m u$ =div$(|Du|^{m-2}Du)$ and $f(u)$ is a non–decreasing continuous function with $f(0) = 0$. We extend this principle to a wide class of singular inequalities involving quasilinear divergence structure elliptic operators, and also consider the converse problem of compact support solutions in exterior domains.

1. Introduction

In a classical paper, Vazquez established a strong maximum principle for the differential inequality

$$\Delta u - f(u) \le 0 \tag{1.1}$$

in a domain D of $\mathbb{R}^n$, where f is a non–decreasing continuous function in $\mathbb{R}$ with $f(0) = 0$. The essential assumption required is that

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty \tag{1.2}$$

where $F(u) = \int_0^u f(s)ds$, the case $f(u) \equiv 0$ on $[0, \tau)$, $\tau > 0$, being included in (1.2) by agreement. More precisely, he showed under these conditions that *any non–negative solution of* (1.1) *in D which vanishes at some point of D must vanish everywhere in D.*

Vazquez also extended this result to the m–Laplace inequality

$$\Delta_m u - f(u) \le 0, \qquad m > 1, \tag{1.3}$$

provided that condition (1.2) is replaced by

$$\int_0^1 \frac{ds}{[F(s)]^{1/m}} = \infty, \tag{1.4}$$

though the details were not fully carried out.

Conditions (1.2) and (1.4) are known to be necessary for the validity of the strong maximum principle for the inequalities (1.1) and (1.3), see respectively [BBC] and [D].

Indeed in Theorem 1.4 of [D] more is proved. Consider the general divergence structure inequality

$$\operatorname{div}\{A(|Du|)Du\} - f(u) \le 0 \tag{1.5}$$

under the ellipticity condition that $\Omega = tA(t)$ is continuously differentiable in $(0, \infty)$ with $\Omega'(t) > 0$ for $t > 0$ and $\Omega(t) \to 0$ as $t \to 0$. Then it was shown that the strong maximum principle *cannot hold* when

$$\int_0^1 \frac{ds}{H^{-1}(F(s))} < \infty, \tag{1.6}$$

where $H = H(t)$ is the strictly increasing function on $[0, \infty)$ defined by $H(0) = 0$ and

$$H(t) = t^2 A(t) - \int_0^t sA(s)ds, \qquad t > 0. \tag{1.7}$$

For example, the case of the m–Laplace operator $A(t) = t^{m-2}$, $m > 1$, yields $H(t) = (m-1)t^m/m$, so that (1.6) becomes exactly the converse of (1.4).

Taking this discussion into account, we are led to the conjecture that *the strong maximum principle should hold for* (1.5) *precisely if*

$$\int_0^1 \frac{ds}{H^{-1}(F(s))} = \infty. \tag{1.8}$$

This conjecture was partially answered in the affirmative in [DST]. Our first main result is to provide a more complete resolution of the conjecture, see Theorem 1 in Section 2. In so doing we simultaneously weaken the ellipticity condition on $\Omega(t)$, requiring only that it be continuous and strictly increasing on $(0, \infty)$ and that it approach zero as $t \to 0$.

For the m–Laplace operator, condition (1.8) reduces to Vazquez's condition (1.4). On the other hand, for the minimal surface operator $A(t) = 1/\sqrt{1+t^2}$ we have $H(t) = 1 - 1/\sqrt{1+t^2} \approx \frac{1}{2}t^2$ as $t \to 0$, so that (1.8) is equivalent to (1.2), that is, the same condition which is required for the Laplace operator.

Inequality (1.5), in the special case of equality, is the Euler–Lagrange equation for the canonical variational problem

$$\delta \int_D \{G(|Du|) + F(u)\}dx = 0,$$

where F, $G \in C^1[0, \infty)$ with $F(0) = G(0) = F'(0) = G'(0) = 0$, and where G is strictly convex in $[0, \infty)$. The relation with (1.5) is attained by defining $A(t) = G'(t)/t$ for $t > 0$. It is worth noting that

$$H(t) = tG'(t) - G(t)$$

is then the Legendre transform of the integrand G. In particular, for a regular integrand, namely whenever $G \in C^2[0,\infty)$ with $G''(t) > 0$ for all $t \geq 0$, we have $H(t) \approx \frac{1}{2}G''(0)t^2$ as $t \to 0$, and condition (1.8) reduces exactly to (1.2), again the same as for the Laplace operator.

In Theorem 2 we consider the case when (1.6) holds. Under this condition, it is shown that *any non-negative solution of the converse inequality*

$$\operatorname{div}\{A(|Du|)Du\} - f(u) \geq 0 \tag{1.9}$$

in the exterior of a ball in $\mathbb{R}^n$*, which approaches* 0 *as* $|x| \to \infty$*, must vanish for all sufficiently large* $|x|$*, that is, must either vanish identically or have compact support.*

In Theorem 3 and its corollary we formalize the result noted above, that (1.8) is a necessary condition for the strong maximum principle to hold for (1.5).

The results described above can be extended to a wider class of differential inequalities by replacing $\operatorname{div}\{A(|Du|)Du\}$ by the more general operator $\operatorname{div}\{a^{ij}(x)A(|Du|)D_j u\}$ and $f(u)$ by $B(x,u,Du)$, where $a^{ij}(x)$ is a positive definite symmetric matrix on D and where B satisfies a condition of the form

$$B(x,u,p) \leq \text{ Const.}\,|p|\,A(|p|) + f(u) \tag{1.10}$$

for $x \in D$, $u \geq 0$ and all $p \in \mathbb{R}^n$ with $|p|$ sufficiently small (reverse the inequality sign for the compact support principle!). *These extensions are the second main purpose of the paper*; see Theorems $1'$ and $2'$ in Section 4.

An important prototype is the equation

$$\Delta_m u - |Du|^p - f(u) = 0, \qquad m > 1,\ p > 0. \tag{1.11}$$

Since $A(t) = t^{m-2}$ for this case, condition (1.10) then requires $p \geq m-1$; that is, the strong maximum principle holds for (1.11) when $p \geq m-1$ and f obeys (1.8). On the other hand, when $p \in (0, m-1)$ the strong maximum principle can fail, even when f obeys (1.8), e.g. the C^1 function $u(x) = C|x|^k$ satisfies

$$\Delta_m u - |Du|^p = 0, \tag{1.12}$$

where

$$k = 1 + \frac{1}{s}, \qquad \frac{1}{C} = k\left[(m-1)\frac{1}{s} + n - 1\right]^{1/s}, \qquad s = m-1-p > 0$$

(for $m = 2$, this example is due to G. Barles, G. Diaz & J.I. Diaz). It is of further interest in connection with this example that the compact support principle fails when $p > m-1$, e.g. the function $u(x) = L|x|^{-l}$ satisfies (1.12) with $l = |1 + 1/s| > 0$ provided that

$$p < \min\big(m, \frac{n(m-1)}{n-1}\big), \qquad \frac{1}{L} = l\left|(m-1)\frac{1}{s} + n - 1\right|^{1/s}.$$

In the next section we state the main hypotheses on the operator A and the nonlinearity f, and give our main results for the canonical models (1.5) and (1.9). Their proofs are contained in Section 3, while in Section 4 we consider the case of fully quasilinear inequalities

$$\mathrm{div}\{a^{ij}(x)A(|Du|)D_j u\} - B(x,u,Du) \le 0 \ \ (\ge 0).$$

It is worth adding that the comparison principles in Section 5 are of interest in themselves, as generalizations of the well known comparison Theorem 10.7 of [GT][1] to degenerate quasilinear equations.

Part I. Canonical Inequalities

2. Main results

Consider the differential inequalities

$$\mathrm{div}\{A(|Du|)Du\} - f(u) \le 0, \qquad u \ge 0, \tag{2.1}$$

and

$$\mathrm{div}\{A(|Du|)Du\} - f(u) \ge 0, \qquad u \ge 0, \tag{2.2}$$

in a domain D, possibly unbounded, of $\mathbb{R}^n$, $n \ge 2$. Here we assume without further mention the following conditions on $A = A(t)$ and $f = f(u)$,

(A1) $A \in C(0,\infty)$,
(A2) $t \mapsto tA(t)$ is strictly increasing in $(0,\infty)$ and $tA(t) \to 0$ as $t \to 0$;
(F1) $f \in C[0,\infty)$,
(F2) $f(0) = 0$ and f is non–decreasing on some interval $[0,\delta)$, $\delta > 0$.

By a *solution* of (2.1) or (2.2) in D we mean a non-negative function $u \in C^1(D)$ which satisfies (2.1) or (2.2) in the distribution sense.

We define $F(u) = \int_0^u f(s)ds$, $u > 0$. Also, with the notation $\Omega(t) = tA(t)$ when $t > 0$, and $\Omega(0) = 0$, we introduce the function

$$H(t) = t\Omega(t) - \int_0^t \Omega(s)ds, \qquad t \ge 0. \tag{2.3}$$

Letting $\Omega^{-1}(\omega)$ be the inverse of the strictly increasing function $\Omega(t)$, then from Stieltjes integration it is easy to see that

$$H(t) = \int_0^{\Omega(t)} \Omega^{-1}(\omega)d\omega, \qquad t \ge 0. \tag{2.4}$$

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[1] Theorem 9.5 in the first edition.

Therefore H is strictly increasing on $[0, \infty)$.

We can now state our first main result.

Theorem 1 (Strong maximum principle). *Suppose*

$$\liminf_{t\to 0} \frac{H(t)}{t\Omega(t)} > 0, \tag{2.5}$$

and either $f(s) \equiv 0$ *for* $s \in [0, \tau)$, $\tau > 0$, *or*

$$\int_0^\delta \frac{ds}{H^{-1}(F(s))} = \infty. \tag{2.6}$$

If u *is a solution of* (2.1) *with* $u(x_0) = 0$ *for some* $x_0 \in D$, *then* $u \equiv 0$ *in* D.

Remarks. Condition (2.5) is perhaps unexpected. It is however automatically satisfied for the canonical example of the m–Laplacian, and for any variational problem with regular integrand G, as follows from the comments in the introduction. At the same time, (2.5) is *not* a consequence of conditions (A1), (A2), as shown by the example

$$A(t) = \left(t \log \frac{1}{t}\right)^{-1} \left[1 + \left(\log \frac{1}{t}\right)^{-1}\right],$$

for which (A1), (A2) are satisfied but not (2.5).

Note added in proof. We have recently shown by using a more delicate version of Lemma 2 (in Section 3) that (2.5) is in fact not needed for Theorem 1.

It is well-known for linear equations that there is a close relation between the strong maximum principle and the standard boundary point lemma. The same relation holds true here, see in particular Corollary 1′ in Section 4.

In the next result we consider the situation when the integral in (2.6) is convergent. Here the appropriate hypothesis is that u "vanishes" at ∞, rather than at some finite point $x_0 \in D$.

Theorem 2 (Compact support principle). *Suppose* $f(s) > 0$ *for* $s \in (0, \delta)$ *and*

$$\int_0^\delta \frac{ds}{H^{-1}(F(s))} < \infty. \tag{2.7}$$

Let D *be unbounded, with* $\{x \in \mathbb{R}^n : |x| > R\} \subset D$ *for some* $R > 0$. *If* u *is a solution of* (2.2) *in* D, *with* $u(x) \to 0$ *as* $|x| \to \infty$, *then there exists* $R_1 \geq R$ *such that* $u(x) \equiv 0$ *for* $|x| > R_1$.

Theorem 2 also applies when f satisfies the alternative conditions

(F1)′ $f \in C(0, \infty)$,
(F2)′ f is a maximal graph with $f(0) = 0$ and $\liminf_{u\to 0} f(u) > 0$ (or $+\infty$);

see the remark following the proof of Theorem 2 in the next section.

Theorem 2 is essentially due to Redheffer [R, Theorem 4], by specializing his results to the matrix $a_{ij} = A(|p|)\delta_{ij} + |p|A'(|p|)p_i p_j/|p|^2$ which arises by expansion of the divergence term in (2.2). This specialization requires, however, two assumptions which are not needed here, first that the operator A be of class $C^1(0,\infty)$, and second, that the solutions in consideration should be of class C^2 at points of D where $Du \neq 0$. [In the proof of Theorem 4 of [R] it is not evident that an appropriate comparison principle can be applied without the further assumption that the nonlinearity f be non–decreasing for small $u > 0$ – that is, for the validity of Theorem 4 of [R] this additional assumption, which is exactly (F2) above, seems to be required as well.] For the special case of the degenerate Laplacian, see also [DH].

If Theorem 2 were an exact analogue of Theorem 1, the conclusion would be that $u \equiv 0$ in D, but this would be incorrect since (2.2) admits non–trivial compact support solutions under assumption (2.7). We formalize this statement in the following

Theorem 3. *Suppose that $f > 0$ in $(0,\delta)$. Let $D_R = \{x \in \mathbb{R}^n \ : \ |x| > R\}$. Then if R is sufficiently large, there is a non–trivial solution of* (2.1) *in D_R, with the equality sign, such that $u(x) \to 0$ as $|x| \to \infty$.*

This result has the important consequence,

Corollary 1. *Condition* (2.6) *is necessary for the strong maximum principle to be valid for* (2.1).

Proof. Suppose (2.6) fails, with $f > 0$ in $(0,\delta)$, and let u be the solution obtained in Theorem 3 for $D = D_R$. By Theorem 2, since (2.7) now holds, u vanishes for $|x| > R_1 \geq R$. But $u \not\equiv 0$, which violates the strong maximum principle. [Another proof of the corollary will be given in the next section.]

Corollary 2. *Suppose* (2.5) *holds. Then condition* (2.7) *is necessary for the compact support principle to be valid for* (2.2).

Proof. Suppose (2.7) fails and let u be the solution obtained in Theorem 3 for $D = D_R$. By Theorem 1, since both (2.5) and (2.6) now hold, it is clear that $u > 0$ in D_R, but this violates the compact support principle.

3. Proofs

3.1 For the proof of Theorem 1 we require several preliminary lemmas.

Lemma 1. (i)*For any constant $\sigma \in [0,\, 1]$ we have*

$$F(\sigma u) \leq \sigma F(u), \qquad u \in [0,\, \delta).$$

(ii)*Let $w = w(r)$ be of class $C^1(r_0, r_1)$ with $w'(r) \geq 0$, and assume that $\Omega \circ w'$ is also of class $C^1(r_0, r_1)$. Then $H \circ w'$ is of class $C^1(r_0, r_1)$, and*

$$\{H(w'(r))\}' = w'(r)\{\Omega(w'(r))\}' \qquad \text{in } (r_0, r_1).$$

To obtain (i), observe that $\sigma f(\sigma u) \le \sigma f(u)$ for $u \in (0, \delta)$, since f is non-decreasing. Integrating this relation from 0 to u yields the result.

On the other hand, (ii) is an immediate consequence of (2.4).

Lemma 2. *Suppose conditions* (2.5) *and* (2.6) *are satisfied. Then for any numbers* k*,* $R > 0$ *and* $\varepsilon \in (0, \delta)$ *the ordinary differential equation*

$$\{H(|v'|)\}' + \frac{k}{r}H(|v'|) - f(v)v' = 0 \tag{3.1}$$

has a C^1 *solution* $v = v(r)$ *in the interval* $\frac{1}{2}R \le r \le R$*, with*

$$v(R) = 0, \qquad v'(R) = -\alpha < 0 \tag{3.2}$$

and

$$0 < v < \varepsilon, \quad -1 < v' < 0 \qquad \textit{in } [R/2,\, R) \tag{3.3}$$

provided α *is sufficiently small.*

Proof. By a standard Schauder fixed point argument (see [FLS, Proposition A1]) there exists a *maximal interval* $[R_0, R]$, $\frac{1}{2}R \le R_0 < R$, for which a C^1 solution $v = v(r)$ of (3.1), (3.2) exists and satisfies

$$0 \le v \le \varepsilon, \quad -1 \le v' \le 0 \qquad \text{in } [R_0, R]. \tag{3.4}$$

Step 1. We assert that $v > 0$ and $|v'| > \alpha$ in $[R_0, R)$. To see this, multiply (3.1) by r^k to get

$$\{r^k H(|v'|)\}' = r^k f(v)v' \le 0 \qquad \text{in } [R_0, R]. \tag{3.5}$$

Therefore by integration we obtain

$$H(|v'(r)|) \ge \left(\frac{R}{r}\right)^k H(\alpha) \ge H(\alpha) > 0,$$

and the claim follows since H is strictly increasing.

Step 2. Clearly from (3.5)

$$\{r^k H(|v'(r)|)\}' - R^k\{F(v(r))\}' \ge 0 \qquad \text{for } r \in [R_0, R].$$

Integrating from r to R yields, since $F(0) = 0$,

$$R^k H(\alpha) - r^k H(|v'(r)|) + R^k F(v(r)) \ge 0,$$

and in turn

$$H(|v'(r)|) \le \left(\frac{R}{r}\right)^k \{F(v(r)) + H(\alpha)\}.$$

Therefore, finally, since $r \ge R_0 \ge \frac{1}{2}R$,

$$|v'(r)| \le H^{-1}\big(2^k\{F(v(r)) + H(\alpha)\}\big) \tag{3.6}$$

in $[R_0, R]$.

Step 3. We complete the proof of the lemma. Consider first the case when $f(u) \equiv 0$ in some interval $[0, \tau)$. Without loss of generality, we can assume that $\varepsilon \le \tau$. Then by (3.6) we get $|v'(r)| \le H^{-1}\big(2^k H(\alpha)\big)$ in $[R_0, R]$. Hence for $\alpha > 0$ sufficiently small, $|v'(r)| < \max\{1,\ 2\varepsilon/R\}$. Therefore $-1 < v' < 0$, $0 < v < \varepsilon$ on $[R_0, R)$, from which we conclude that $R_0 = R/2$ for the maximal interval in which (3.4) holds. This completes the proof of the lemma for the first case.

The principal case, when $f(u) > 0$ for $0 < u < \delta$, is more delicate. Let $\sigma = 2^{-(k+1)}$; without loss of generality we can assume that

$$\varepsilon < \sigma\delta, \qquad F(\varepsilon) < \sigma H(1). \tag{3.7}$$

Let $\alpha_0 > 0$ be fixed such that

$$H(\alpha_0) < F(\varepsilon). \tag{3.8}$$

For $\alpha < \alpha_0$ we define $\hat{\varepsilon}(\alpha) = F^{-1}(H(\alpha))$. Clearly $\hat{\varepsilon}(\alpha) < \varepsilon$ by (3.8). Now for any fixed $\alpha \in (0, \alpha_0]$ and for any $r \in [R_0, R]$ we have either

$$v(r) > \hat{\varepsilon}(\alpha) \qquad \text{or} \qquad v(r) \le \hat{\varepsilon}(\alpha) < \varepsilon.$$

In the first case we claim that if α is sufficiently small then also $v(r) < \varepsilon$. Indeed in this case it is clear that $F(v(r)) > F(\hat{\varepsilon}(\alpha)) = H(\alpha)$, so that by (3.6)

$$|v'(r)| < H^{-1}\left(\frac{1}{\sigma}F(v(r))\right).$$

Clearly there exists $r_0 \in (r, R)$ where $v(r_0) = \hat{\varepsilon}(\alpha)$. Integrating from r_0 to r then yields

$$\int_{\hat{\varepsilon}(\alpha)}^{v(r)} \frac{ds}{H^{-1}(F(s)/\sigma)} < \frac{R}{2};$$

making the substitution $s = \sigma t$ and using Lemma 1(i), we then get (since $t \le \varepsilon/\sigma < \delta$ when $r \in [R_0,\ R]$)

$$\int_{\hat{\varepsilon}(\alpha)/\sigma}^{v(r)/\sigma} \frac{dt}{H^{-1}(F(t))} < 2^k R.$$

The claim now follows from (2.6) and the fact that $\hat{\varepsilon}(\alpha) \to 0$ as $\alpha \to 0$. In summary, whenever α is sufficiently small we have

$$v(r) < \varepsilon \qquad \text{for } r \in [R_0, R];$$

also by (3.6)–(3.8)

$$|v'(r)| \le H^{-1}\big(2^k\{F(\varepsilon) + H(\alpha_0)\}\big) < H^{-1}\big(2^k\{\sigma H(1) + \sigma H(1)\}\big) = 1.$$

Consequently $R_0 = R/2$ and the lemma is proved.

Lemma 3 (Weak comparison principle). *Let u and v be respective solutions of* (2.1) *and* (2.2) *in a bounded domain D. Suppose also that u and v are continuous in $\overline{D}$, with $v < \delta$ in D and $u \ge v$ on ∂D. Then $u \ge v$ in D.*

Proof. Let $w = u - v$ in $\overline{D}$. If the conclusion fails, then there exists a point $x_1 \in D$ such that $w(x_1) < 0$. Fix $\varepsilon > 0$ so small that $w(x_1) + \varepsilon < 0$. Consequently, since $w \ge 0$ on ∂D it follows that the function $w_\varepsilon = \min\{w + \varepsilon,\ 0\}$ is non–positive and has compact support in D. By the distribution meaning of solutions, taking the Lipschitzian function w_ε as test function, we get

$$\int_D \{A(|Du|)Du - A(|Dv|)Dv\} Dw_\varepsilon \le \int_D \{f(v) - f(u)\} w_\varepsilon. \tag{3.9}$$

The left hand side of (3.9) is positive due to the strict monotonicity of $tA(t)$ and the fact that $Dw_\varepsilon \equiv Dw \not\equiv 0$ when $w + \varepsilon < 0$, while otherwise $Dw_\varepsilon = 0$ (a.e.).

Moreover, when $w + \varepsilon < 0$ there holds $0 \le u < v - \varepsilon$ $(< \delta$; hence $f(v) - f(u) \ge 0$ since $f(s)$ is non–decreasing for $s < \delta$. Thus the right hand side of (3.9) is non–positive, a contradiction. This completes the proof of the lemma.

Remark. Lemma 3 is essentially the result of Theorem 10.7 of [GT], with **A** independent of z and B independent of p, and with the differentiability of **A** replaced by a strict convexity condition; see also Theorem 4′ in Section 5.

Now we are ready to prove Theorem 1. We first show that the function $v(x) = v(r)$, $r = |x|$, where v is given by Lemma 2, satisfies the differential inequality (2.2) in $E_R = \{x \in \mathbb{R}^n \ :\ R/2 \le |x| \le R\}$. This is a consequence of the calculation:

$$\begin{aligned} v'\big[\operatorname{div}\big\{A(|Dv|)Dv\big\} - f(v)\big] &= v'\big[\operatorname{div}\big\{A(|v'|)v'x/r\big\} - f(v)\big] \\ &= |v'|\big[\big\{\Omega(|v'|)\big\}' + \frac{(n-1)}{r}\Omega(|v'|)\big] - v'f(v) \\ &= \{H(|v'|)\}' + \frac{(n-1)}{r}|v'|\,\Omega(|v'|) - v'f(v) \\ &\le \{H(|v'|)\}' + kH(|v'|)/r - v'f(v) = 0, \end{aligned}$$

where Lemma 1 has been used at the third step, while at the last step we have applied (2.5) and the fact that $|v'| < 1$, with the constant

$$k = (n-1) \sup_{|t|\le 1} \frac{t\Omega(t)}{H(t)}.$$

Therefore, since $v' < 0$, the function v is a solution of (2.2) in E_R.

The rest of the proof is exactly the same as in the standard demonstration of the strong maximum principle (see [GT, proof of Theorem 3.5 on page 35), since the comparison function v satisfies the conditions, see [GT, proof of Lemma 3.4 on page 34]:

(i) $v > 0$ in E_R,
(ii) $v = 0$ when $|x| = R$,
(iii) $\partial v/\partial \nu = v' < 0$ when $|x| = R$, where ν is the outer normal to ∂E_R,
(iv) $v < \varepsilon$ when $|x| = R/2$,

where ε, $R > 0$ can be taken arbitrarily small and the origin of coordinates can be chosen arbitrarily in D. Note that the use of the weak maximum principle (Corollary 3.2 of [GT]) is here replaced by application of Lemma 3.

This completes the proof of Theorem 1.

3.2. Proof of Theorem 2. By (2.7) we can define

$$C = \int_0^\delta \frac{ds}{H^{-1}(F(s))} < \infty.$$

Introduce $w = w(r)$, $0 \le r < C$, by

$$r = \int_{w(r)}^\delta \frac{ds}{H^{-1}(F(s))}. \tag{3.10}$$

Differentiation gives

$$-\frac{w'(r)}{H^{-1}(F(w(r))} = 1 \qquad \text{for } 0 \le r < C,$$

that is, w is of class $C^1[0,C)$, with $0 < w \le \delta$, $w' < 0$, and $H(|w'|) = F(w)$. Hence also, from Lemma 1,

$$-\{\Omega(|w'|)\}' = f(w). \tag{3.11}$$

Obviously $w(r) \to 0$, $w'(r) \to 0$ as $r \to C$. Therefore, by defining $w(r) \equiv 0$ for $r \ge C$, it is clear that w becomes a C^1 solution of (3.11) in $[0,\infty)$.

Now let u be the solution of (2.2) in D given in the statement of the theorem, with $u(x) \to 0$ as $|x| \to \infty$. Clearly there exists $R_0 \ge R$ such that $u(x) < \delta$ if $|x| \ge R_0$. For any $x \in D_0 = \{x \in \mathbb{R}^n \ : \ |x| > R_0\}$, define $v(x) = w(|x| - R_0)$. Consequently, for $x \in D_0$, and $r = |x|$, we have

$$\operatorname{div}\{A(|Dv|)Dv\} - f(v) = -\{\Omega(|v'|)\}' - \frac{(n-1)}{r}\Omega(|v'|) - f(v) \leq 0$$

in view of (3.11) and the fact that $\Omega \geq 0$. Since $0 \leq u(x) < \delta = v(x)$ on ∂D_0, and since $u(x)$, $v(x) \to 0$ as $|x| \to \infty$, we can apply the comparison principle Lemma 3 (with the roles of u and v being interchanged) to obtain $0 \leq u(x) \leq v(x)$ in D_0. In particular $u(x) = 0$ when $|x| \geq R_1 = R_0 + C$, as required.

Remarks. 1. The last sentence of the proof gives an a priori estimate for the support of u.

2. Theorem 2 is closely related to the results in [R], see also the remarks after the statement of Theorem 2 in Section 2. The proof we have given is in fact not different in its underlying ideas from those in [BBC, CEF, DH, R, V], the principal improvements here being the direct approach, the generality of the solution class, and the clarification of the method.

3. When conditions (F1)′, (F2)′ are assumed, rather than (F1), (F2), we can transform the vertical segment of f at $u = 0$ into a linear segment with finite slope, thus arriving at a function $\overline{f} \leq f$ satisfying (F1) and (F2). But then every solution of (2.2) remains a solution of (2.2) with f replaced by $\overline{f}$, and the result of Theorem 2 continues to apply.[2]

A second proof of Corollary 1. Suppose (2.6) fails, that is (2.7) holds. We can then introduce the function $w = w(r)$, defined on $[0, \infty)$, as in the proof of Theorem 2 For any $x \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \ : \ x_n > 0\}$, let $u(x) = w(x_n)$. By (3.11), u is obviously a solution of (2.1), *with the equality sign*, in the domain $D = R^n_+$. Clearly $u(0, \ldots, 0, C) = w(C) = 0$ and at the same time $u \not\equiv 0$ in D. Hence the strong maximum principle fails.

This result also yields a direct and simple answer to the unique continuation question for the equation $\operatorname{div}\{A(|Du|)Du\} - f(u) = 0$, that is, the function $u(x) = w(x_n)$ shows that a solution in a domain D may vanish in a subdomain without vanishing throughout D. Another counterexample to unique continuation was given in the proof of Corollary 1 in Section 2.

3.3. Proof of Theorem 3. Let the function $\tilde{f}$ be defined on $[0, \infty)$ by

$$\tilde{f}(u) = \begin{cases} f(u), & 0 \leq u \leq \delta, \\ -u + f(\delta) + \delta, & \delta \leq u \leq \beta, \\ \ u + \tilde{f}(\beta) - \beta, & \ \ u \geq \beta, \end{cases}$$

where $\beta > \delta$ is chosen so that

$$\int_0^\beta \tilde{f}(s)ds = 0.$$

[2] A similar argument can be used also for maximal monotone graphs f, see [V].

Clearly $\tilde{f}(\gamma) = 0$, where $\gamma = \beta - \tilde{f}(\beta) > \beta$.

Case 1. Suppose that $H(t) \to \infty$ as $t \to \infty$. Then from Theorem A of [FLS, page 180], there exists a radially symmetric non–negative, non–trivial C^1 solution of

$$\mathrm{div}\{A(|Du|)Du\} - \tilde{f}(u) = 0 \qquad \text{in } \mathbb{R}^n$$

such that $u(x) \to 0$ as $|x| \to \infty$. This solution moreover has the property that $u'(r) \leq 0$ for $r \geq 0$, and $u(0) \in (\beta, \gamma)$.

Let $R > 0$ be so large that $u(x) = \delta$ when $|x| = R$. Thus $u(x) \leq \delta$ when $|x| \geq R$. Consequently u is a solution of

$$\mathrm{div}\{A(|Du|)Du\} - f(u) = 0 \qquad \text{in } D_R,$$

and u does not vanish identically in D_R. This completes the proof of Case 1.

Case 2. Assume next that $H(\infty) < \infty$. Then the above argument requires the further condition

$$h = \max_{[0,\beta]} \tilde{F}(u) - \tilde{F}(\gamma) < H(\infty) \tag{3.12}$$

(see case (c) of Theorem A in [FLS]). It is not hard to see that

$$h = 2\int_0^\delta f(s)ds + f(\delta)^2.$$

Therefore, in order to verify (3.12) it is enough, if necessary, to replace δ by a sufficiently small number $\tilde{\delta} \in (0, \delta)$ in the construction of $\tilde{f}$. The rest of the proof is now the same as in case 1.

Remark. It is the authors' belief that Theorem 3 can be proved more simply and directly, without applying the work of [FLS].

Part II. Fully Quasilinear Inequalities

4. The strong maximum principle and the compact support principle

4.1. Let D be a domain in $\mathbb{R}^n$. Let $\{a^{ij}(x)\}$, $i, j = 1, \cdots, n$, be a continuously differentiable, symmetric coefficient matrix on D, which is uniformly elliptic in the sense that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad x \in D,\ \xi \in \mathbb{R}^n,$$

for some positive number λ. Moreover, let $B(x, u, p)$ be a continuous function on $D \times \mathbb{R}_0^+ \times \mathbb{R}^n$.

We shall suppose without further mention that the operator $A = A(t)$ satisfies the following strengthened versions of (A1), (A2), namely

(A1)$'$ $A \in C^1(0, \infty)$,
(A2)$'$ $\Omega'(t) > 0$ for $t > 0$, and $\Omega(t) \to 0$ as $t \to 0$,
(A3)$'$ condition (2.5) holds;

we also continue to assume that the nonlinearity f obeys (F1) and (F2).

The following direct extension of Theorem 1 now holds.

Theorem 1′ (Strong maximum principle). *Consider the differential inequality*[3]

$$\operatorname{div}\{a^{ij}(x)A(|Du|)D_j u\} - B(x,u,Du) \le 0, \qquad u \ge 0, \quad x \in D. \tag{4.1}$$

Assume that there exists a constant $\kappa > 0$ such that

$$B(x,u,p) \le \kappa\Omega(|p|) + f(u) \tag{4.2}$$

for $x \in D$, $u \ge 0$, and all $p \in \mathbb{R}^n$ with $|p| < 1$. Suppose finally that either $f(s) \equiv 0$ for $s \in [0,\tau)$, $\tau > 0$, or else (2.6) *holds.*

If u is a C^1 solution of (4.1) *with $u(x_0) = 0$ for some $x_0 \in D$, then $u \equiv 0$ in D.*

The semilinear case $A(t) = 1$ was treated by Vazquez [V, Theorem 4]. (Note that in [V] the condition (C4) should include the assumption $d(x) > 0$.) For earlier work on linear inequalities, see [C]; and for non–singular quasilinear inequalities with $B(x,u,p)$ Lipschitz continuous in u, p, see [S].

Proof. The proof is similar to that of Theorem 1. We first show that for any origin O in D the function $v(x) = v(r)$, $r = |x|$, where v is given by Lemma 2, satisfies the differential inequality

$$\operatorname{div}\{a^{ij}(x)A(|Du|)D_j u\} - \kappa\Omega(|Du|) - f(u) \ge 0, \qquad u \ge 0, \tag{4.3}$$

in $E_R = \{x \in \mathbb{R}^n \ : \ R/2 \le |x| \le R\}$. Let

$$\Lambda = \text{max eigenvalue of } \{a^{ij}(x)\} \text{ in } E_R, \qquad a = \max\, |D_i\, a^{ij}(x)| \text{ in } E_R.$$

It is easy to see that

$$D_i\big(a^{ij}(x)\frac{x_j}{r}\big) = \big(D_i\, a^{ij}(x)\big)\frac{x_j}{r} + \frac{a^{ij}}{r}\big(\delta_{ij} - \frac{x_i x_j}{r^2}\big),$$

so

$$\max_{E_R} |D_i\big(a^{ij}(x)\frac{x_j}{r}\big)| \le a + \frac{n-1}{r}\Lambda. \tag{4.4}$$

By a trivial change of scale we can suppose without loss of generality that $\lambda = 1$. Then calculations similar to those in the proof of Theorem 1 yield (since $-1 < v' < 0$)

$$\begin{aligned} v'\big[&\operatorname{div}\big\{a^{ij}(x)A(|Dv|)D_j v\big\} - \kappa\Omega(|Dv|) - f(v)\big] \\ &\le a^{ij}(x)\frac{x_i x_j}{r^2}|v'|\{\Omega(|v'|)\}' + \big(a + \kappa + \frac{n-1}{r}\Lambda\big)|v'|\,\Omega(|v'|) - v' f(v) \\ &\le a^{ij}(x)\frac{x_i x_j}{r^2}\big[\{H(|v'|)\}' + k[\Lambda + R(a+\kappa)]H(|v'|)/r - v' f(v)\big] = 0, \end{aligned}$$

[3] By div $\mathbf{A}^i$ we mean $D_i(\mathbf{A}^i)$, with the obvious summation.

where at the last steps we have used (A3)$'$ and Lemma 2 (replacing k with $k[\Lambda+R(a+\kappa)]$). Therefore, again since $v'<0$, the function v is a solution of (4.3) in E_R.

We next require a comparison lemma corresponding to Lemma 3, but applying to the more general inequality (4.1).

Lemma 4 (Comparison principle). *Let u and v be respectively solutions of* (4.1) *and* (4.3) *in a bounded domain D. Suppose that $|Du|+|Dv|>0$ in D; that u and v are continuous in $\overline{D}$; and that*

$$v<\delta \quad \text{in } D, \qquad u\geq v \quad \text{on } \partial D.$$

Then $u\geq v$ in D.

Lemma 4 has been stated in the context of the inequalities (4.1) and (4.3), but the result immediately extends to the inequalities treated in Theorem 10.7 of [GT]; see Section 5. The main point of Lemma 4 is that if $|Du|+|Dv|>0$ in D, then just as for Lemma 3 *it is not necessary to have ellipticity at the value $p=0$*. We defer the proof of Lemma 4 until Section 5.

The rest of the proof of Theorem 1$'$ is now the same as before, the only change being that at the last step we rely on Lemma 4 instead of Lemma 3.

The following boundary lemma is immediate as a by-product of Theorem 1$'$. It will be important for the proof of Theorem 2$'$ below.

Corollary 1$'$ (Boundary lemma). *Let $x_0\in\partial D$ and suppose that D satisfies an interior sphere condition at x_0. Assume that the conditions of Theorem 1$'$ hold.*

If u is a C^1 solution of (4.1) *in $\overline{D}$, with $u>0$ in D and $u=0$ at x_0, then $\partial u/\partial\nu<0$ at x_0, where ν is the outer normal to ∂D at x_0.*

Proof. We proceed essentially as in Theorem 1$'$. By the interior sphere condition there exist $y\in D$ and $R>0$ such that $B=B_R(y)\subset D$ and $x_0\in\partial B$. Let v be the solution of (4.3) given in Lemma 2 and put $w(x)=v(|x-y|)$. Then as in the proof of Theorem 1$'$ it follows that

$$u(x)\geq w(x) \quad \text{in } B_R(y)\backslash B_{R/2}(y)$$

provided that ε is sufficiently small. This completes the proof of the corollary, since $\partial w/\partial\nu=v'(R)<0$.

4.2. We next consider the corresponding compact support principle, when condition (2.7) rather than (2.6) is satisfied. Let D be unbounded, with $\{x\in\mathbb{R}^n : |x|>R\}\subset D$ for some $R>0$, and consider the inequality

$$\text{(4.5)} \qquad \operatorname{div}\{a^{ij}(x)A(|Du|)D_ju\}-B(x,u,Du)\geq 0, \qquad u\geq 0,$$

in D, where for some constant $\kappa>0$,

$$\text{(4.6)} \qquad B(x,u,p)\geq -\kappa\Omega(|p|)+f(u)$$

for $x \in D$, $u \geq 0$, and all $p \in \mathbb{R}^n$ with $|p| < 1$.

Theorem 2′ (Compact support principle). *Let* (4.6) *hold. Suppose* $f(s) > 0$ *for* $s \in (0, \delta)$ *and let* (2.7) *be satisfied. If* u *is a solution of* (4.5) *in* D *with* $u(x) \to 0$ *as* $|x| \to \infty$*, then there exists* $R_1 \geq R$ *such that* $u(x) \equiv 0$ *for* $|x| > R_1$*.*

Remark. For the importance of the lower bound (4.6), see the counterexample after the proof.

Proof. The proof is almost the same as that of Theorem 2, except that we need to reconstruct the comparison function v. For simplicity, we shall assume as before that $\lambda = 1$.

Fix $\sigma \in (0, 1)$. By using Lemma 1(i) and the strict monotonicity of H^{-1}, we get

$$\int_0^\delta \frac{ds}{H^{-1}(\sigma F(s))} \leq \int_0^\delta \frac{ds}{H^{-1}(F(\sigma s))} = \frac{1}{\sigma} \int_0^{\delta\sigma} \frac{ds}{H^{-1}(F(s))} < \infty$$

by (2.7). It follows that $\gamma \in (0, \delta)$ can be chosen so that $F(\gamma) \leq H(1)$ and

$$0 < C = \int_0^\gamma \frac{ds}{H^{-1}(\sigma F(s))} \leq 1. \tag{4.7}$$

Introduce $w = w(r)$, $0 \leq r < C$, by

$$r = \int_{w(r)}^\gamma \frac{ds}{H^{-1}(\sigma F(s))}.$$

Differentiation gives

$$-\frac{w'(r)}{H^{-1}(\sigma F(w(r)))} = 1 \qquad \text{for } 0 \leq r < C,$$

that is, w is of class $C^1[0, C)$, with $0 < w \leq \gamma$, $-1 < w' < 0$, $w'' > 0$, and $H(|w'|) = \sigma F(w)$. Hence also, from Lemma 1,

$$-\{\Omega(|w'|)\}' = \sigma f(w). \tag{4.8}$$

Obviously $w(r) \to 0$, $w'(r) \to 0$ as $r \to C$. Therefore, by defining $w(r) \equiv 0$ for $r \geq C$, it is clear that w becomes a C^1 solution of (4.8) in $[0, \infty)$.

For any $R_0 > 0$ and $D_0 = \{x \in \mathbb{R}^n \ : \ |x| > R_0\}$, let us define $v(x) = w(|x| - R_0)$. Then, for $x \in D_0$ and $r = |x|$ we have, as in the earlier calculations in the proof of Theorem 1′,

$$\begin{aligned}
&\operatorname{div}\big\{a^{ij}(x)A(|Dv|)D_j v\big\} + \kappa\Omega(|v'|) - f(v) \\
&\qquad \leq - a^{ij}(x)\frac{x_i x_j}{r^2}\{\Omega(|v'|)\}' + \left(a + \kappa - \frac{n-1}{r}\right)\Omega(|v'|) - f(v) \\
&\qquad \leq a^{ij}(x)\frac{x_i x_j}{r^2}\sigma f(w) + (a+\kappa)k\frac{H(|v'|)}{|v'|} - f(v) \\
&\qquad \leq KH(|v'|)/|v'| - (1 - \Lambda\sigma)f(v);
\end{aligned} \tag{4.9}$$

the steps in obtaining (4.9) are as follows:

First line: when $|x| > R_0 + C$ we have $v \equiv 0$ so there is nothing to show, while for $|x| \le R_0 + C$ there exist constants $a = a_0 > 0$ and $\Lambda = \Lambda_0 > 0$ such that

$$|D_i a^{ij}(x))| \le a, \qquad a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2;$$

and finally, $a^{ij}(x)(\delta_{ij} - x_i x_j/r^2) \ge n - 1$ (since $\lambda = 1$).

Second line: use (A3)$'$, (4.8), and the fact that $\Omega \ge 0$;

Third line: use (4.6) and put $K = (a + \kappa)k$.

Next we want to show that, for $\sigma > 0$ small enough,

$$KH(|v'|) \le (1 - \Lambda\sigma)f(v)|v'|. \tag{4.10}$$

Since f is non-decreasing for $s < \delta$, there holds $F(v) \le vf(v)$. Moreover, since $v' < 0$ and $v'' > 0$,

$$v(|x|) = \int_{C+R_0}^{|x|} v'(s)ds \le C|v'(|x|)| \le |v'(|x|)|,$$

since $C \le 1$. By taking $\sigma = 1/(K + \Lambda)$, it now follows that[4]

$$\frac{K\sigma}{1 - \Lambda\sigma} \cdot \frac{F(v)}{f(v)} \le v \le |v'|.$$

Therefore

$$KH(|v'|) = K\sigma F(v) \le (1 - \Lambda\sigma)f(v)|v'|,$$

which is (4.10). Combining (4.9) and (4.10) we have finally

$$\operatorname{div}\{A(|Dv|)Dv\} + \kappa\Omega(|v'|) - f(v) \le KH(|v'|)/|v'| - (1 - \Lambda\sigma)f(v) \le 0. \tag{4.11}$$

By assumption, since $u(x) \to 0$ as $|x| \to \infty$, we can choose $R_0 > R$ so that $u(x) < \gamma$ in D_0, where γ is given in (4.7). We wish to show that $u \le v$ as in the proof of Theorem 2, where v is the comparison function above, satisfying (4.11). For this purpose it is not possible to resort directly to Lemma 4, since $Dv \equiv 0$ for large $|x|$ while Du is unrestricted. Accordingly we use an indirect argument.

Define $\omega = v - u$ in D_0. We claim that $\omega \ge 0$. If this is not the case, then

$$\omega_0 = \inf_{D_0} \omega < 0$$

and we shall reach a contradiction. Note first that $\omega = \gamma - u > 0$ when $|x| = R_0$, and that $\omega(x) \to 0$ as $|x| \to \infty$; hence the infimum of ω must be attained at some point y in D_0.

[4]Clearly $\sigma < 1$ since $\Lambda \ge \lambda = 1$ and $K > 0$.

Let $R_1 = R_0 + C$, where C is given by (4.7), and define

$$\hat{D} = \{R_0 < |x| < R_1\}, \qquad D_1 = \{|x| > R_1\}.$$

Then $D_0 = \hat{D} \cup \partial D_1 \cup D_1$, so exactly the following three cases can occur:

1. The infimum of ω is attained in D_1.

2. The infimum of ω is not attained in D_1, but is reached at a point on ∂D_1.

3. The infimum of ω is not attained in $\overline{D}_1$, but is reached in $\hat{D}$.

In Case 1, let the infimum be attained at x_0. For x in D_1, define $z(x) = -u(x) - \omega_0$. Then since $v \equiv 0$ in D_1, we see that $z = \omega - \omega_0 \geq 0$ has a zero minimum at x_0. Moreover,

$$\operatorname{div}\{a^{ij}(x)A(|Dz|)D_j z\} - B_1(x, z, Dz) \leq 0, \qquad z \geq 0,$$

in D_1, where (using (4.6) and the fact that $f(u) > 0$ when $u < \gamma$)

$$B_1(x, z, Dz)) = -B(x, u, Du) \leq \kappa\Omega(|Du|) - f(u) \leq \kappa\Omega(|Dz|).$$

That is, B_1 satisfies (4.2) with $f \equiv 0$. Hence by the strong maximum principle (Theorem $1'$) applied to the domain D_1 we obtain $z \equiv 0$. Thus $u \equiv -\omega_0 > 0$ in D_1, which is impossible since $u(x) \to 0$ as $|x| \to \infty$.

In Case 2, let the infimum of ω be reached at x_0 on ∂D_1. In this case, obviously $z > 0$ in D_1 while $z = 0$ at x_0 (we can of course consider z as a C^1 function on $\overline{D}_1$). Then, since D_1 clearly satisfies an interior sphere condition at x_0, the boundary lemma (Corollary $1'$) gives $\partial z/\partial\nu < 0$ at x_0. But this is also impossible, because $Dz = D\omega = 0$ at x_0.

In Case 3, necessarily $v - u = \omega > \omega_0$ in $\overline{D}_1$, while as noted earlier $v - u > 0$ when $|x| = R_0$. Thus $v - u > \omega_0$ on the boundary of the domain $\hat{D}$ and of course $u < \delta$ and $Dv \neq 0$ in $\hat{D}$. This is exactly the case of Lemma 4 for $D = \hat{D}$ (with the roles of u and v interchanged). The conclusion is that $v - u > \omega_0$ in $\hat{D}$ (not $v - u \geq 0$!). But this contradicts the condition of Case 3 that $\omega = v - u$ attains its infimum ω_0 in $\hat{D}$

We have thus shown that all three cases lead to a contradiction. Consequently $\omega \geq 0$ in D_0, that is $v \geq u$. In turn, $u \equiv 0$ for $|x| > R_1$, which completes the proof of the theorem.

Here we give a counterexample showing the importance of the lower bound condition (4.6). Consider the inequality

$$\Delta_m u + |Du|^p - u^q \geq 0, \qquad m > 1, \;\; p,\, q > 0. \tag{4.12}$$

Clearly, conditions (2.7) and (4.6) are satisfied if and only if $p \geq m-1$ and $q < m-1$. The compact support principle then holds for (4.12). On the other hand, for any $p \in (0, m-1)$ we can take $p < q < m - 1$. One easily checks that (4.12) then has positive solutions $u = \text{const.}\,|x|^{-l}$ on $\{|x| > R\}$ for l and R large. Hence the compact support principle fails even though condition (2.7) is fulfilled!

4.3. The inequalities (4.1) and (4.5) do not involve the variable u in the divergence term, a restriction which in some cases can be overcome. Confining the discussion for simplicity to the strong maximum principle, we consider the inequalities

$$\operatorname{div}\{a^{ij}(x,u)A(|Du|)D_j u\} - B(x,u,Du) \le 0, \qquad u \ge 0, \tag{4.13}$$

and

$$\operatorname{div}\{a^{ij}(x)c(u)A(|Du|)D_j u\} - B(x,u,Du) \le 0, \qquad u \ge 0. \tag{4.14}$$

For (4.13), assume that $a^{ij}(x,u)$ is continuously differentiable in x and u, with

$$a^{ij}(x,u)\xi_i\xi_j \ge \lambda|\xi|^2, \qquad x \in D,\ u \in [0,\infty)\ \xi \in \mathbb{R}^n,$$

for some positive number λ, and that $B(x,u,p) \le f(u)$, i.e., $\kappa = 0$ in (4.2). The proof of Theorem 1′ then carries over essentially unchanged up to the application of Lemma 4. This lemma, however, is no longer applicable since the variable u appears in the divergence term. On the other hand, we can use Theorem 10.7 (ii) of [GT] to obtain a corresponding comparison theorem which can be applied (see the remark after Theorem 4′ in the Section 5). Here it is crucial that $\hat{B}(x,u,p)$ (which in the present case is to be identified with $f(u)$) is independent of the gradient variable p. In summary, when (2.5) and (2.6) hold, one obtains the strong maximum principle for (4.13) exactly as for (4.1).

For (4.14), assume that $c(u)$ is positive and continuously differentiable for $u \in [0,\delta)$, while $B(x,u,p)$ continues to satisfy (4.2). We rewrite (4.14) in the form

$$\operatorname{div}\{a^{ij}(x)A(|Du|)D_j u\} - \tilde{B}(x,u,Du) \le 0, \qquad u \ge 0. \tag{4.15}$$

where, for some constant $\tilde{\kappa} \ge \kappa$,

$$\begin{aligned}\tilde{B}(x,u,p) =& -a^{ij}(x)\left(\frac{c'(u)}{c(u)}\right)A(|p|)p_i p_j + \frac{B(x,u,p)}{c(u)}\\ \le& \tilde{\kappa}\Omega(|p|) + \text{const.}\, f(u)\end{aligned}$$

for $0 \le u < \delta$ and $|p| < 1$. Clearly Theorem 1′ applies to (4.15), so again the strong maximum principle is valid when (2.5) and (2.6) are satisfied.

5. A comparison theorem for singular divergence form operators

Consider the differential inequalities

$$\operatorname{div}\{\hat{A}(x,Du)\} - \hat{B}(x,u,Du) \le 0, \qquad u \ge 0, \tag{5.1}$$

$$\operatorname{div}\{\hat{A}(x,Dv)\} - \hat{B}(x,v,Dv) \ge 0, \qquad v \ge 0, \tag{5.2}$$

in a possibly unbounded domain $D \subset \mathbb{R}^n$. Let the vector function

$$\hat{A}(x,p): \; D \times \mathbb{R}^n \to \mathbb{R}^n$$

be continuous in $D \times \mathbb{R}^n$ and continuously differentiable with respect to p for $p \neq 0$ in $\mathbb{R}^n$; and let

$$\hat{B}(x,z,p): \; D \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

be continuous in $D \times \mathbb{R} \times \mathbb{R}^n$ and continuously differentiable with respect to p for $p \neq 0$ in $\mathbb{R}^n$. Suppose moreover that $\hat{A}$ is uniformly elliptic in the sense that the matrix $\{D_j \hat{A}^i(x,p)\}$ is positive definite for $x \in D$ and $p \neq 0$ in $\mathbb{R}^n$. Finally assume that $\hat{B}(x,z,p)$ is non-decreasing in the variable z. Then the following comparison principle holds.

Theorem 4 (Comparison principle). *Let u and v be respective solutions of* (5.1) *and* (5.2) *in D. Suppose that u and v are continuous in $\overline{D}$, that $|Du| + |Dv| > 0$ in D, and that $u \geq v$ on ∂D.*[5] *Then $u \geq v$ in D.*

This is exactly Theorem 10.7 (i) of [GT] with the exception that the functions $\hat{A}$ and $\hat{B}$ are allowed to be singular at $p = 0$, this being compensated by the additional condition $|Du| + |Dv| > 0$ in D. [We have written $\hat{A}$, $\hat{B}$ here, rather A, B as in [GT], in order to avoid confusion with earlier notation in the paper.]

Proof. Suppose for contradiction that the theorem is false. Put $w(x) = u(x) - v(x)$, whence

$$\bar{\varepsilon} = -\inf_{x \in D} w(x) > 0.$$

By the boundary condition, clearly there exists $x_0 \in D$ such that $w(x_0) = -\bar{\varepsilon}$.

For $\varepsilon \in (0, \bar{\varepsilon})$, plainly $w_\varepsilon = \min\,(w + \varepsilon, \, 0)$ has compact support in D. Then, as in the proof of Lemma 3, one has

$$\begin{aligned} \int_D \{\hat{A}(x,Du) - \hat{A}(x,Dv)\} Dw_\varepsilon &\leq \int_\Sigma \{\hat{B}(x,v,Dv) - \hat{B}(x,u,Du)\} w_\varepsilon \\ &\leq \int_\Sigma \{\hat{B}(x,u,Dv) - \hat{B}(x,u,Du)\} w_\varepsilon, \end{aligned} \tag{5.3}$$

where

$$\Sigma = \Sigma_\varepsilon = \text{supp } w_\varepsilon$$

is a compact subset of D, and where at the last step of (5.3) we have used the facts that $w_\varepsilon \leq 0$ and $u \leq v$ in Σ, and that $\hat{B}$ is non-decreasing in the variable z.

[5] If D is unbounded, the boundary condition is understood to include the limit relation

$$\liminf_{|x| \to \infty} \{u(x) - v(x)\} \geq 0$$

.

As in [GT, Proof of Theorem 10.7], define $u_t = tv + (1-t)u$, $t \in [0,1]$, and

$$\hat{a}^{ij}(x) = \int_0^1 D_{p_j} A^i(x, Du_t)\, dt, \qquad \hat{c}^i(x) = \int_0^1 D_{p_i} B(x, u(x), Du_t)\, dt.$$

Then (5.3) can be written

$$\int_D \hat{a}^{ij}(x) D_i w_\varepsilon D_j w_\varepsilon \leq \int_\Sigma \hat{c}^i(x) D_i w_\varepsilon w_\varepsilon. \tag{5.4}$$

At this point we wish to proceed as in the proof of Theorem 8.1 in [GT], but this requires further preliminary argument since $\hat{A}$ and $\hat{B}$ are now allowed to be singular at $p = 0$.

We claim that if $\varepsilon \in (0, \bar{\varepsilon})$ is sufficiently close to $\bar{\varepsilon}$ then

$$|Du_t| \geq \text{const.} > 0 \quad \text{in } \Sigma. \tag{5.5}$$

To prove (5.5), let

$$d = \frac{1}{6} \min\{|Du| + |Dv|\} \quad \text{in } \hat{\Sigma} = \operatorname{supp} w_{\bar{\varepsilon}/2} \subset D.$$

Note particularly that $d\,(>0)$ is independent of ε.

Obviously $Du - Dv = Dw = 0$ on the closed subset $E = \{x \in D : \ w(x) = -\bar{\varepsilon}\}$ of Σ. Moreover, distance$(E, \partial\Sigma) \to 0$ as $\varepsilon \to \bar{\varepsilon}$. Hence by continuity, $|Du - Dv| < d$ in Σ provided ε is suitably near $\bar{\varepsilon}$. In particular, for such values of ε we find easily that

$$\min\{|Du|, |Dv|\} \geq 2d$$

and so

$$|Du_t| = |tDu + (1-t)Dv| \geq |Dv| - t|Du - Dv| \geq d > 0 \quad \text{in } \Sigma,$$

which is just (5.5).

This being shown, we now have by the principal hypotheses on the functions $\hat{A}$ and $\hat{B}$,

$$\hat{a}^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \xi \in \mathbb{R}^n; \qquad |c^i(x)| \leq \lambda^{-1}$$

for all $x \in \Sigma$ and all ε suitably near $\bar{\varepsilon}$, where λ is an appropriate positive constant (independent of ε). Hence from (5.4) it follows that

$$\int_D |Dw_\varepsilon|^2 \leq \lambda^{-2} \int_\Sigma |Dw_\varepsilon||w_\varepsilon|. \tag{5.6}$$

Let $\Gamma = \Gamma_\varepsilon = \{\bar{\varepsilon} - \varepsilon < w_\varepsilon < 0\}$. Then $Dw_\varepsilon = 0$ a.e. on $\Sigma \setminus \Gamma$, so the integral on the right side of (5.6) can equally be taken over the set Γ.

Using the Hölder and Sobolev inequalities, we now obtain (see the proof of Theorem 8.1 in [GT], 2nd edition)

$$|\Gamma| \geq C\lambda^{2n} \tag{5.7}$$

where C depends only on the dimension n. On the other hand, $\Gamma \to \emptyset$ as $\varepsilon \to \bar{\varepsilon}$, a contradiction to (5.7). This completes the proof.

When $\hat{B}$ is independent of p, Theorem 4 continues to hold without assuming that $\hat{A}$ is differentiable with respect to p. This gives the following generalization of Lemma 3.

Theorem 4′ (Comparison principle). *Let the hypotheses of Theorem* 4 *hold, except for the condition that* $\hat{A}$ *is differentiable with respect to* p. *Suppose that* $\hat{B}$ *is independent of* p *and that* $\hat{A}$ *is strictly monotone, that is*

$$\{\hat{A}(x,p) - \hat{A}(x,q)\} \cdot (p-q) > 0, \qquad \textit{when } p \neq q.$$

Then $u \geq v$ *in* D.

This follows at once from (5.3), exactly as in the proof of Lemma 3.

Remarks. A related case, again under the condition that $\hat{B}$ is independent of p but allowing $\hat{A}$ to depend on u (continuously differentiably when $p \neq 0$), continues to yield the result of Theorem 4; see Theorem 10.7(ii) of [GT] together with (5.5).

If in Theorems 4 and 4′ we add the hypothesis that $v < \delta$, then the monotonicity of $\hat{B}$ is needed only in the interval $0 < z < \delta$; see the proof of Lemma 3.

We can now give the

Proof of Lemma 4. First observe that, for some constant $\kappa > 0$, the functions u and v respectively satisfy the inequalities

$$\operatorname{div}\{a^{ij}(x)A(|Du|)D_j u\} - \kappa\Omega(|Du|) - f(u) \leq 0,$$

and

$$\operatorname{div}\{a^{ij}(x)A(|Dv|)D_j v\} - \kappa\Omega(|Dv|) - f(v) \geq 0$$

in D. [The first inequality is a consequence of (4.1), (4.2) together with the continuity of $B(x,z,p)$.] Thus we can apply Theorem 4 (together with the remark above). Lemma 4 then follows immediately from the identifications

$$\hat{A}^i(x,p) = a^{ik}(x)A(|p|)p_k, \qquad \hat{B}(x,z,p) = \kappa\Omega(|p|) + f(z),$$

provided we show that the matrix $\{D_{p_j}\hat{A}^i(x,p)\}$ is positive definite for $p \neq 0$. But

$$D_{p_j}\hat{A}^i(x,p) = a^{ik}(x)b^{kj}(p),$$

where

$$b^{kj}(p) = A(|p|)\delta_{kj} + |p|\, A'(|p|)\, \frac{p_i p_k}{|p|^2}, \qquad p \neq 0.$$

The matrix $\{b^{kj}(p)\}$ has eigenvalues $A(|p|)$ (repeated $n-1$ times) and $\Omega'(|p|)$. By assumption (A2)$'$ we have $\Omega'(|p|) > 0$ for $p \neq 0$, while also

$$A(|p|) = \Omega(|p|)/|p| > 0, \qquad \text{for } p \neq 0$$

since Ω is increasing and $\Omega(0) = 0$. Hence $\{b^{ij}\}$ is positive definite for $p \neq 0$. Because $\{a^{ij}(x)\}$ is assumed positive definite in D, it now follows that $\{D_{p_j} \hat{A}^i(x,p)\}$ is positive definite for $x \in D$ and $p \neq 0$, completing the proof.

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Department of Mathematics, University of Perugia

Department of Mathematics, University of Minnesota

Department of Mathematics, University of Alabama at Birmingham.