

Lectures on Stability

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1. Introduction

In this short course we shall present a new reformulation of the direct method for stability of nonlinear ordinary differential systems, often called *Lyapunov's second method*. The main idea is to construct appropriate auxiliary functions to study the stability without recourse to the explicit form of solutions, or exame of the linear approximation of the nonlinear system under consideration.

Our theory is so elastic to cover not only very singular second order ordinary differential systems, but also second order partial differential systems of hyperbolic and parabolic type. Our results for stability are new also for scalar equations of canonical form and even in the linear case. For this reason, in order to present the main ideas in clarity, in the first part of the course we shall discuss the simplest typical prototype we cover, namely

$$(1.1) \quad u'' + f(t, u) = Q(t, u, u'),$$

where the time t varies in any interval I of the form $[T, \infty)$ and $T > 0$ is determined by the specific problem under consideration. Actually we shall discuss here only subcases of (1.1) and refer to the main papers [26–31] for the asymptotic stability of solutions of very general quasi-variational systems.

The system (1.1) may be considered as the motion equation for a holonomic dynamical system with N degrees of freedom whose Lagrangian \mathcal{L} is defined by an action energy $T = G(u, p) = \frac{1}{2}|p|^2$ and a potential $U = F(t, u)$, being $f = \partial F/\partial u$, and whose dynamics are governed by a general nonlinear damping $Q = Q(t, u, p)$.

We remark that quasi-variational systems (1.1) can be written in the so called Hamiltonian form

$$(1.2) \quad \begin{cases} u' = v, \\ v' = -f(t, u) + Q(t, u, v). \end{cases}$$

Because of the great importance of the system (1.2) in applications, we discussed in [29] the derivation of the Hamiltonian form from general quasi-variational Lagrangian systems.

The main point of the new approach is that we can consider the non-autonomous case of (1.1) in a very simple way which allows to cover the two delicate limit cases in which the damping term is neither far from zero nor bounded above as a function of time. Since there are too many concepts of stability, we decided to present here only the very

few ones we consider important and allow to clarify the presentation by significant simple examples.

The typical problem of stability in ordinary differential systems is the following: we have a system, say, of type (1.1) governing an unknown function $u : I = [T, \infty) \rightarrow \mathbb{R}^N$, with the property that $u_0 \equiv 0$ is a solution of (1.1), or at least there is some solution $u_0 = u_0(t)$ of (1.1) such that $\lim_{t \rightarrow \infty} u_0(t) = 0$. We ask whether *all* other solutions of (1.1) approach zero as $t \rightarrow \infty$. In this case we say, by definition, that the rest state $u_0 \equiv 0$ is *globally asymptotically stable*, or that $u_0 \equiv 0$ is a *global attractor for the system* (1.1).

This is a very interesting feature of many problems in control theory or nuclear reactor dynamics in which the rest state $u_0 \equiv 0$ is known to be globally asymptotically stable, in other words the region of asymptotic stability is the whole space \mathbb{R}^N .

But it is obvious, after a moment of reflection, that there are a number of interesting closely related questions: perhaps not *all* solutions of (1.1), but *only all solutions of* (1.1) *which remain bounded as* $t \rightarrow \infty$, or only solutions *starting near enough* zero approach zero as $t \rightarrow \infty$.

Or we might simply ask whether *all* solutions are merely *bounded* as $t \rightarrow \infty$, called *Lagrange's stability for the system* (1.1), or whether all solutions which are suitably small at $t = T$ remain in a ball of any given radius ε and center at 0 for all $t \rightarrow \infty$, called *simple stability*.

More precisely, for a canonical initial value problem

$$(1.3) \quad \begin{cases} y' = \varphi(t, y), & t \in I = [T, \infty), \\ y(T) = y_0, \end{cases}$$

we say that $u_0 \equiv 0$ is *simply stable* if for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for each $y_0 \in \mathbb{R}^M$ with $|y_0| \leq \delta$ every solution y of (1.3) can be continued in I and $|y(t)| \leq \varepsilon$ for all $t \in I$.

We say that $u_0 \equiv 0$ is *asymptotically stable* for (1.3) if $u_0 \equiv 0$ is simply stable and there is $\delta_0 > 0$ such that $|y_0| \leq \delta_0$ implies that all solutions y of (1.3) have the property that $\lim_{t \rightarrow \infty} y(t) = 0$.

The definition of *global asymptotic stability* was already stated rigorously and we shall essentially concentrate the course in this last one.

The classical example of simple stability but not asymptotic stability of $u_0 \equiv 0$ is given by the oscillator equation

$$(1.4) \quad u'' + u = 0, \quad N = 1,$$

which, in form (1.2) and consequently (1.3), becomes

$$\begin{cases} y'_1 = y_2, & y = (y_1, y_2) \\ y'_2 = -y_1, & M = 2. \end{cases}$$

Of course here $u_0 \equiv 0$ is a solution of (1.4) and *all* solutions are bounded. Hence we have Lagrange's stability. Furthermore fixed $\varepsilon > 0$ for $\delta = \varepsilon/\sqrt{2}$ we see that if $|u(T)| \leq \delta$ and $|u'(T)| \leq \delta$, then for all $t \in I$

$$(1.5) \quad [u(T)]^2 + [u'(T)]^2 \leq 2\delta^2.$$

Indeed, our problem is *conservative* by (1.4), being

$$[u^2 + u'^2]' = 2uu' + 2u'u'' = 2u'(u + u'') = 0.$$

Hence by (1.5), in particular,

$$|u(t)| \leq \sqrt{2}\delta \leq \varepsilon \quad \text{in } I,$$

as required. Every solution of (1.4) is of the form

$$u(t) = C_1 \cos t + C_2 \sin t$$

so that it is clear that *only* the solution $u_0 \equiv 0$ approaches zero as $t \rightarrow \infty$, see Theorem 5.1 of [26]. Consequently $u_0 \equiv 0$ is simply stable but *not* asymptotically stable for (1.4).

The rest state $u_0 \equiv 0$ can be made asymptotically stable provided that equation (1.4) is equipped with an additional *damping term*, more precisely (1.4) is replaced by

$$(1.6) \quad u'' + 2\varepsilon u' + u = 0, \quad \varepsilon > 0.$$

This linear parametric equation admits solutions of the form

$$u(t) = e^{-\varepsilon t} \cdot \begin{cases} (C_1 \cos \sqrt{1 - \varepsilon^2}t + C_2 \sin \sqrt{1 - \varepsilon^2}t) & \text{if } 0 < \varepsilon < 1, \\ (C_1 + C_2 t) & \text{if } \varepsilon = 1, \\ (C_1 e^{\sqrt{\varepsilon^2 - 1}t} + C_2 e^{-\sqrt{\varepsilon^2 - 1}t}) & \text{if } \varepsilon > 1. \end{cases}$$

Consequently *all* solutions of (1.6) are bounded – Lagrange's stability – and $u_0 \equiv 0$, which is a solution of (1.6), is not only simply stable but clearly asymptotically stable since $u(t) \rightarrow 0$ as $t \rightarrow \infty$ for every $C_1, C_2 \in \mathbb{R}$ and every $\varepsilon > 0$. Actually $u_0 \equiv 0$ is globally asymptotically stable for (1.6), and also $u'(t) \rightarrow 0$ as $t \rightarrow \infty$.

For the Van der Pol equation

$$(1.7) \quad u'' + \varepsilon(1 - u^2)u' + u = 0, \quad \varepsilon > 0,$$

the trivial solution $u_0 \equiv 0$ is asymptotically stable but not globally asymptotically stable since $\delta_0 = \sqrt{\varepsilon}$. Indeed rewrite (1.7) in the form

$$[u' + \varepsilon(u - \frac{1}{3}u^3)]' + u = 0,$$

so that (1.7) is equivalent to the system

$$(1.8) \quad \begin{cases} y_1' = y_2 - \varepsilon(y_1 - \frac{1}{3}y_1^3) = \varphi_1(y) \\ y_2' = -y_1 = \varphi_2(y). \end{cases}$$

The *energy function*

$$V(y) = V(y_1, y_2) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 = \frac{1}{2}|y|^2$$

has total derivative with respect to the system (1.8) given by

$$\begin{aligned} V'(y) &: \equiv DV(y) \cdot \varphi(y) = \sum_{i=1}^2 \frac{\partial V}{\partial y_i}(y) \varphi_i(y) = y_1[y_2 - \varepsilon(y_1 - \frac{1}{3}y_1^3)] - y_1y_2 \\ &= -\varepsilon y_1^2(1 - \frac{1}{3}y_1^2). \end{aligned}$$

Thus $V' \leq 0$ only if $|y_1| \leq \sqrt{\varepsilon}$, namely $|u| \leq \sqrt{\varepsilon}$.

This is the main idea of the direct method of stability in which $V'(y)$ can be computed directly from the differential system (1.8) without any knowledge of its solutions.

The last simple three examples can be treated directly and so with the usual Lyapunov's second method. We shall next present our modification of this classical method.

Our prototype of (1.1) is the system

$$(1.9) \quad \begin{aligned} u'' + h(t)|u|^\alpha |u'|^\beta u' + \tau|u|^{q-2}u &= 0, \\ \alpha \geq 0, \quad \beta > -1, \quad q > 1, \end{aligned}$$

which comes from the choice $Q(t, u, p) = h(t)|u|^\alpha |p|^\beta p$ in (1.1). Here h is a non-negative function, in accordance with the natural requirement

$$(Q(t, u, p), p) \leq 0 \quad \text{for all arguments,}$$

which can oscillate between zero and infinity. It is this possibility which makes the problem non-classical, even in the scalar case $N = 1$, and even when $q = 2$, because it prevents the use of standard stability theory based on the location of the eigenvalues of the linearized problem – in fact their real parts are for (1.9) not necessarily bounded away from zero or infinity.

The classical Lyapunov theory does not apply directly since the natural candidate for a Lyapunov function for (1.9) is

$$V(y) = \frac{\tau}{q}|y_1|^q + \frac{1}{2}|y_2|^2$$

has total derivative

$$V'(y) = -h(t)|y_1|^\alpha |y_2|^{\beta+2},$$

and clearly the function

$$W(t, y) = -h(t)|y_1|^\alpha|y_2|^{\beta+2}$$

is not negative definite in the required sense, namely $W(t, 0) = 0$ and $W(t, y) < 0$ at every other point y of a neighborhood U of the origin $y = 0$, and for every $t \in I$. Thus both standard Lyapunov theory for stability and its more recent generalizations involving two Lyapunov functions fail [11, Theorems 1.1.2 and 1.1.4].

To further stress the delicacy of the problem, consider the scalar equation

$$(1.10) \quad u'' + h(t)|u|u' + u = 0,$$

which corresponds to the choice $q = 2$, $\alpha = \tau = 1$, $\beta = 0$ and $N = 1$ in (1.9).

If h is too small, say $h \in L^1(I)$, then there are solutions of (1.10) which are bounded but do not tend to zero as $t \rightarrow \infty$. To see this, assume, without loss of generality,

$$\int_I h(s)ds \leq \frac{1}{4}$$

(if this is not the case, we can make the change of variable $v = u/4\|h\|_1$, since $h \in L^1(I)$). Consider the initial value problem $|u'(T)|^2 + |u(T)|^2 = 1$. If we multiply (1.10) by u' and integrate, we obtain

$$|u'(T)|^2 + |u(T)|^2 = -2 \int_T^t h(s)|u(s)| \cdot |u'(s)|^2 ds + 1,$$

from which we immediately derive using also (1.11) that

$$\frac{1}{2} \leq |u'(t)|^2 + |u(t)|^2 \leq 1 \quad \text{in } I.$$

Even more, by Theorem 5.1 in [26] we can actually conclude that no solution other than the trivial one $u_0 \equiv 0$ approaches zero as $t \rightarrow \infty$.

On the other hand, if h is too large, say $h(t) \geq t^{1+\varepsilon}$ for some $\varepsilon > 0$, then there are equations for which the rest state is not a global attractor. Indeed, when $T > 0$ and

$$h(t) = \varepsilon^{-1}t^{1+\varepsilon} + (\varepsilon + 1)(t + t^{1-\varepsilon})^{-1}$$

the equation (1.10) admits the solution

$$u(t) = 1 + t^{-\varepsilon},$$

which is bounded but does not tend to zero as $t \rightarrow \infty$.

Therefore, both for h small, due to resonant behavior, and for large h , due to overdamping, the rest state of (1.9) loses asymptotic stability.

As these examples suggest, the concept of asymptotic stability should be defined with respect to *bounded* solutions rather than with respect to *all* solutions. The motivation for this point of view is that the concepts of boundedness and of stability are essentially different and accordingly should be treated separately.

The determination of appropriate conditions on the damping term Q in (1.1) which cause that all the bounded solutions decay to zero as $t \rightarrow \infty$ will be the topic of the second part of this course. This problem has been studied by several authors in recent years, see in particular the work of Artstein & Infante [1], Ballieu & Peiffer [2], Burton [4], Levin & Nohel [19], Smith [36] and Thurston & Wong [37] for $N = 1$, and Duffin [7], Leoni [14–16], Pucci & Serrin [26–34], Salvadori [35] and Yoshizawa [38] for systems, $N \geq 1$.

2. Remarks on Lyapunov stability

The principal ingredients of Lyapunov's direct method consist first in the choice of an appropriate Lyapunov inequality corresponding to the first order differential system

$$(2.1) \quad y' = \varphi(t, y) \quad t \in I = [T, \infty),$$

where, for simplicity, we assume that $\varphi : I \times \mathbb{R}^M \rightarrow \mathbb{R}^M$ is continuous. More generally, we could assume that φ is of Carathéodory type and solutions of (2.1) are simply to be continuous on I and absolutely continuous on compact subintervals of $I^0 = (T, \infty)$, satisfying (2.1) a.e. in I .

We say that $V : I \times \mathbb{R}^M \rightarrow \mathbb{R}$ is a *Lyapunov function* for (2.1) if V is of class C^1 , with the corresponding definition of derivative of V with respect to the system (2.1), namely

$$V'_\varphi \equiv V_t + V_y \cdot \varphi : I \times \mathbb{R}^M \rightarrow \mathbb{R},$$

satisfying appropriate differential inequalities.

In particular, if $y = y(t)$ is a solution of (2.1) then

$$V'_\varphi(t, y(t)) = \frac{d}{dt}V(t, y(t)).$$

The concept of *wedge* plays a second fundamental role in the theory. We say that $W : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, $\mathbb{R}_0^+ = [0, \infty)$, is a wedge if W is a continuous non-decreasing function with $W(0) = 0$ and $W(s) > 0$ for $s > 0$.

We also denote by $AC(I)$ the family of all continuous functions $P : I \rightarrow \mathbb{R}$ which are absolutely continuous on every compact subinterval of I^0 .

The next results are based on the following trivial fact.

Proposition. *Let $P : I \rightarrow \mathbb{R}$ be a bounded below function of class $AC(I)$ such that*

$$P'(t) \leq -p(t) \quad \text{a.e. in } I,$$

where $p \in L_{\text{loc}}^1(I)$. Then

$$\limsup_{t \rightarrow \infty} \int_T^t p(s) ds < \infty.$$

Proof. Since $P \in AC(I)$ then

$$P(t) - P(T) \leq - \int_T^t p(s) ds,$$

so that

$$\int_T^t p(s)ds \leq P(T) - P(t).$$

This immediately gives the assertion since P is bounded below.

All functions which will be denoted by the W letter, possibly changing from one case to the next without specific mention, are wedges in what follows. In the sequel V , \hat{V} , etc. . . . , will always denote Lyapunov functions for the system (2.1), and B a corresponding given open set of \mathbb{R}^M containing 0.

The first basic result we prove is (see Theorem 1.1.2 of [11]).

Theorem A. *Suppose that V is bounded above and below by wedges, that is*

$$(2.2) \quad W_1(|y|) \leq V(t, y) \leq W_2(|y|)$$

and that

$$(2.3) \quad V'_\varphi(t, y) \leq -W(|y|)$$

for all $(t, y) \in I \times B$. Then every solution $y = y(t)$ of (2.1) which ultimately lies in B approaches zero as $t \rightarrow \infty$, namely

$$(2.4) \quad \lim_{t \rightarrow \infty} y(t) = 0.$$

Proof. Let $y = y(t)$ be a solution of (2.1) which ultimately lies in B . Then by (2.3) it is easily seen that $V(\cdot, y(\cdot))$ is non-increasing in I and by (2.2)₁ there is a non-negative number ℓ such that

$$\lim_{t \rightarrow \infty} V(t, y(t)) = \ell.$$

If $\ell = 0$ then (2.4) holds by (2.2)₁ and the definition of wedge. If $\ell > 0$ then by (2.2)₂ and (2.3) there follows

$$\frac{dV}{dt}(t, y(t)) \leq -c, \quad \text{where } c = \text{const.} > 0.$$

Taking $p(t) \equiv c$ in the proposition now yields a contradiction. Hence the case $\ell > 0$ cannot occur.

Note that the conclusion of Theorem A as well as those of our next results can be written alternately saying that any solution $y = y(t)$ of (2.1) which enters in a *positively invariant set* $\Omega \subset B$ of (2.1) must approach zero as $t \rightarrow \infty$. A set Ω is said to be *positively invariant for the system* (2.1) if every solution of (2.1) starting in Ω remains in Ω for all future time.

The customary assumptions $V(t, 0) \equiv 0$ and $V(t, y) \geq 0$, which follow from (2.2), are replaced by the next weaker condition (2.9).

Theorem 1. *Suppose that*

$$(2.5) \quad \hat{V}(t, y) \geq 0$$

$$(2.6) \quad V'_\varphi(t, y) \leq \psi(t) - W(\hat{V}(t, y))k(t)$$

for all $(t, y) \in I \times B$, where

$$(2.7) \quad k \geq 0, \quad \psi, k \in L^1_{\text{loc}}(I), \quad k \notin L^1(I).$$

Assume also that

$$(2.8) \quad \liminf_{t \rightarrow \infty} \frac{\int_T^t \psi(s) ds}{\int_T^t k(s) ds} \leq 0.$$

Let $y = y(t)$ be a solution of (2.1) which ultimately lies in B and also satisfies

$$(2.9) \quad \liminf_{t \rightarrow \infty} V(t, y(t)) > -\infty.$$

Then

$$(2.10) \quad \liminf_{t \rightarrow \infty} \hat{V}(t, y(t)) = 0.$$

Remarks. It is important to note the relation between the canonical conclusion (2.4) of Theorem A and (2.10) of Theorem 1. Indeed, if to the assumptions of Theorem 1 we add that

$$(2.11) \quad \lim_{t \rightarrow \infty} \hat{V}(t, y(t)) \text{ exists,}$$

as in the standard Lyapunov theory where the function $\hat{V}(\cdot, \cdot)$ is decreasing in I , then obviously conclusion (2.10) can be strengthened to

$$\lim_{t \rightarrow \infty} \hat{V}(t, y(t)) = 0.$$

Moreover, if (2.5) is replaced by

$$(2.2)_1 \quad \hat{V}(t, y(t)) \geq W_1(|y(t)|)$$

for some wedge W_1 , then, even more the canonical conclusion (2.4) of Theorem A holds.

Consequently Theorem A is an immediate corollary of Theorem 1 with

$$\psi \equiv 0, \quad k \equiv 1, \quad \hat{V} \equiv V.$$

In particular (2.2)₂ and (2.3) then imply (2.6), while (2.11) is a consequence of (2.2)₁ and (2.3), finally (2.8) is trivially satisfied.

In applying Theorem 1 to particular stability questions, the functions k and V must of course be chosen judiciously, with V generally depending on k , see [26].

Proof of Theorem 1. We argue by contradiction and assume the conclusion false. Then by (2.5) there is a solution $y = y(t)$ of (2.1), which ultimately lies in B , and a number $\ell > 0$ such that

$$\liminf_{t \rightarrow \infty} \hat{V}(t, y(t)) > \ell.$$

By (2.6)

$$(2.12) \quad \frac{d}{dt} V(t, y(t)) \leq \psi(t) - W(\ell)k(t)$$

for $t \geq T_1$ and T_1 sufficiently large. By (2.9) the function $P(t) = V(t, y(t))$ is bounded below on I and clearly of class $AC(I)$. Moreover $p(t) = -\psi(t) + W(\ell)k(t)$ is of class $L^1_{\text{loc}}(I)$ by (2.7). Consequently (2.12) and our proposition yield

$$(2.13) \quad \limsup_{t \rightarrow \infty} \int_{T_1}^t p(s) ds < \infty.$$

On the other hand, by (2.8) there is a sequence $t_j \nearrow \infty$ such that for all j

$$\int_T^{t_j} \psi(s) ds \leq \frac{1}{2} W(\ell) \int_T^{t_j} k(s) ds.$$

Therefore

$$\int_T^{t_j} p(s) ds = - \int_T^{t_j} \psi(s) ds + W(\ell) \int_T^{t_j} k(s) ds \geq \frac{1}{2} W(\ell) \int_T^{t_j} k(s) ds$$

which diverges to ∞ as $j \rightarrow \infty$ since $k \notin L^1(I)$ by (2.7). This contradicts (2.13) and completes the proof.

Remarks. For the several corollaries and related results of independent interest we refer to the original paper [30]. We note, however, that (2.8) is automatic when $\psi \in L^1(I)$ since $k \notin L^1(I)$. If, in addition, $\hat{V} \equiv V$ then (2.9) is a direct consequence of (2.5). Moreover by (2.6) it is then clear that $V(t, y(t))$ approaches a limit as $t \rightarrow \infty$. Hence the conclusion (2.10) can be strengthened to the form

$$(2.14) \quad \lim_{t \rightarrow \infty} V(t, y(t)) = 0.$$

The conclusion of Theorem 1 remains valid when (2.8) is weakened by

$$\liminf_{t \rightarrow \infty} \int_T^t \psi(s) \exp\left(-\int_t^s k(r) dr\right) ds \leq 0,$$

provided that (2.9) is strengthened to (2.14).

We refer to the main paper [30] for further general abstract results. Here we present only one corollary, very simple to state, which enclose some earlier results.

Theorem 2. (simple version of Theorem 3 of [30]) *Assume that*

$$(2.15) \quad \begin{aligned} & \hat{V}(t, y) \geq 0 \\ & V'_\varphi(t, y) \leq -W(\hat{V}(t, y))k(t) + \psi(t) \end{aligned}$$

for all $(t, y) \in I \times B$, where

$$(2.16) \quad k \geq 0, \quad k \in L^1_{\text{loc}}(I), \quad \psi \in L^1(I).$$

Suppose also that there exists $\eta > 0$ such that

$$(2.17) \quad \hat{V}'_\varphi(t, y) \leq \hat{\psi}(t) \quad (\text{or } -\hat{V}'_\varphi(t, y) \leq \hat{\psi}(t))$$

whenever $\hat{V}(t, y) \leq \eta$, where

$$(2.18) \quad \hat{\psi} \in L^1_{\text{loc}}(I), \quad \int_J \hat{\psi}(s)ds \leq \hat{W}(|J|), \quad \int_J k(s)ds \geq \tilde{W}(|J|)$$

for all intervals $J \subset I$ with $|J|$ sufficiently small.

Let $y = y(t)$ be a solution of (2.1) which ultimately lies in B and for which

$$(2.19) \quad \liminf_{t \rightarrow \infty} V(t, y(t)) > -\infty.$$

Then

$$\lim_{t \rightarrow \infty} \hat{V}(t, y(t)) = 0.$$

Proof. Assume for contradiction that the conclusion fails. Then, since $\hat{V}(t, y) \geq 0$, there exists $\ell > 0$ and a sequence $t_j \nearrow \infty$ such that

$$\hat{V}(t_j, y(t_j)) > \ell \quad \text{for all } j.$$

We can assume of course that $\ell < \eta$. Now fix $\beta > 0$ so that $\hat{W}(\beta) < \ell/2$ and (2.18) holds whenever $|J| \leq \beta$. By refinement of the sequence (t_j) , if necessary, we can suppose moreover that $t_{j+1} - t_j > \beta$ for all j .

Assume first that (2.17)₁ holds. We then claim that

$$(2.20) \quad \hat{V}(t, y(t)) \geq \ell/2 \quad \text{for } t \in [t_j - \beta, t_j], \quad j = 1, 2, \dots$$

Indeed, let τ_j be the smallest $\tau \in [t_j - \beta, t_j)$ such that (2.20) holds in $[\tau, t_j]$ so that $\hat{V}(\tau_j, y(\tau_j)) = \ell/2 < \eta$ while $\hat{V}(t_j, y(t_j)) > \ell$. Consequently there is $\sigma_j \in (\tau_j, t_j]$ such that $\hat{V}(t, y(t)) \leq \eta$ in $[\tau_j, \sigma_j]$ and $\hat{V}(\sigma_j, y(\sigma_j)) \geq \ell$. Then by (2.17)₁ for all $t \in [\tau_j, \sigma_j]$ it results

$$\hat{V}(t, y(t)) \geq \hat{V}(\sigma_j, y(\sigma_j)) - \int_t^{\sigma_j} \hat{\psi}(s)ds > \ell - \ell/2 = \ell/2,$$

by the choice of β . Therefore $\tau_j = t_j - \beta$ and the claim is proved.

Now by (2.20) and (2.15) for all t sufficiently large

$$\frac{d}{dt}V(t, y(t)) \leq \begin{cases} -W(\ell/2)k(t) + \psi(t), & \text{if } t \in [t_j - \beta, t_j], \\ \psi(t), & \text{otherwise.} \end{cases}$$

We denote by $-p(t)$, $t \in I$, the right hand side of this inequality. By (2.19) it is clear that the $AC(I)$ function $P(t) = V(t, y(t))$ is bounded below on I and by the main proposition we conclude that

$$(2.21) \quad \limsup_{t \rightarrow \infty} \int_T^t p(s) ds < \infty.$$

On the other hand,

$$\begin{aligned} \int_T^{t_j} p(s) ds &= - \int_T^{t_j} \psi(s) ds + W(\ell/2) \sum_{i=1}^j \int_{t_i - \beta}^{t_i} k(s) ds \\ &\geq - \int_T^{t_j} \psi(s) ds + W(\ell/2) \sum_{i=1}^j \tilde{W}(\beta) \\ &= - \int_T^{t_j} \psi(s) ds + jW(\ell/2)\tilde{W}(\beta) \\ &\longrightarrow \infty \quad \text{as } j \rightarrow \infty, \end{aligned}$$

since $\psi \in L^1(I)$ by (2.16). This contradicts (2.21) and completes the proof when (2.17)₁ holds.

The alternative case (2.17)₂ is treated in the same way replacing the main intervals $[t_j - \beta, t_j]$ by $[t_j, t_j + \beta]$.

Remarks. 1. When $\hat{V}(t, y) = |y|^2$ then condition (2.17) reduces to

$$y \cdot \varphi(t, y) \leq \hat{\psi}(t) \quad (\text{or } -y \cdot \varphi(t, y) \leq \hat{\psi}(t))$$

for all $t \in I$ and $|y| \leq \eta$.

2. Salvadori's Theorem 1.1.4 [11] is the special case of the simplified Theorem 2 with

$$k \equiv 1, \quad \psi \equiv 0, \quad \hat{\psi} \equiv M \quad \text{and} \quad V(t, y) \geq 0.$$

In this subcase the conditions (2.19), (2.16) and (2.18) are automatic.

3. The simplified Theorem 2 is almost exactly Theorem 1 of [38], with several minor exceptions. In [38] a weaker condition is assumed for ψ than $\psi \in L^1(I)$ required in (2.16).

It seems, however, that this leads to a gap in the proof (see [38], p.1160) because γ need not be in $L^1[t_1, \infty)$. Second Yoshizawa replaces (2.17) by an abstract condition (d) whose only purpose is to obtain directly the claim (2.20) without the help of (2.17). Finally, he formulates his result with respect to the stability of a general closed subset E of B , as we shall briefly indicate below, rather than with respect to the behavior of $\hat{V}(t, y(t))$ as $t \rightarrow \infty$. More precisely his conclusion becomes exactly the assertion that

$$\text{dist}(y(t), E) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In this context the attractor $y_0 \equiv 0$ is replaced by E . In particular the usual condition

$$(2.22) \quad \hat{V}(t, y) \geq W_1(|y|)$$

is modified to the weaker form

$$(2.23) \quad \hat{V}(t, y) \geq W_1(d), \quad d = \text{dist}(y, E), \quad (t, y) \in I \times B$$

(see also Theorem 1 and its remarks).

We just state a generalization of the Matrosov theorem [11, Theorem 1.4.3].

Let (E_j) be a decreasing sequence of closed sets with property that

$$(2.24) \quad y(t) \in E_j \text{ for all } t \geq T_j \text{ implies } y(t) \in E_{j+1} \text{ for all } t \geq T_{j+1}$$

for any solution $y = y(t)$ of (2.1). Assume also that

$$\bigcap_1^{\infty} E_j = \{0\}.$$

Then every solution of (2.1) which ultimately lies in E_j approaches zero as $t \rightarrow \infty$.

To verify condition (2.24), in the application of this result, it is enough to use an appropriate Lyapunov inequality. Theorems 1–2 and their corollaries, provided that (2.22) is replaced by (2.23), can serve for this purpose.

4. From the original abstract Theorem 3 of [30] we derive a series of corollaries of independent interest one of which is essentially Theorem 2 of [38].

3. Examples

In this section we shall present some easy but significant applications that can be treated by the previous set of arguments and results. As we already pointed out in the preceding section, we presented there only the simplest results of [30] so that the examples

we cover here are not so general as they could have been using Theorem 2 of [30] that we decided to omit in this course.

Consider the canonical system

$$(3.1) \quad u'' + h(t)|u|^\alpha |u'|^\beta u' + \tau|u|^{q-2}u = 0,$$

where h is a non-negative continuous function defined on $I = [T, \infty)$, $T > 0$, where α, β, τ, q are real parameters which satisfy the following inequalities

$$\alpha \geq 0, \quad \beta > -1, \quad \tau > 0, \quad q > 1.$$

From Theorem 5.1 of [26] we know that if $h \in L^1(I)$ then $u_0 \equiv 0$ is not globally asymptotically stable, actually there are no solutions of (3.1) except $u_0 \equiv 0$ which approach a limit in \mathbb{R}^N as $t \rightarrow \infty$. Consequently we must assume, necessarily, that

$$h \notin L^1(I).$$

The function $f(u) = \tau|u|^{q-2}u$, $q > 1$, in (3.1) can be replaced by any continuous restoring force derivable from a potential F , that is,

$$(f(u), u) > 0 \quad \text{for } u \neq 0 \quad \text{and} \quad f = \nabla F$$

(where the last requirement is automatic in the scalar case $N = 1$), satisfying natural mild algebraic growth conditions. Hence, using the ideas of this section, we are able to cover also the typical example in the standard stability theory which is the model of a damped pendulum equation,

$$u'' + u' + f(u) = 0, \quad f(u) = \sin u.$$

Furthermore the celebrated Liénard equation is a special case of our prototype when we replace $g(u) = |u|^\alpha$, $\alpha \geq 0$, by any continuous function g such that $g(u) > 0$ for $u \neq 0$.

In Section 1 we already noted the delicacy of the stability problem for the presence of the term $h = h(t)$ in (3.1), even in its linear scalar subcase, namely when $N = 1$, $\alpha = \tau = 1$, $\beta = 0$ and $q = 2$.

Let us discuss (3.1) first when $\alpha = 0$ so that the system (3.1) can be written in the notation of Section 2, as

$$y' = \varphi(t, y), \quad y \in \mathbb{R}^{2N},$$

or, being $y = (y_1, y_2)$, $\varphi = (\varphi_1, \varphi_2)$,

$$(3.2) \quad \begin{cases} y_1' = y_2 = \varphi_1 \\ y_2' = -h(t)|y_2|^\beta y_2 - \tau|y_1|^{q-2}y_1 = \varphi_2. \end{cases}$$

We shall apply Theorem 1 of Section 2 to (3.2) and shall take as a first Lyapunov function

$$\hat{V}(t, y) = \hat{V}(y) = \frac{\tau}{q}|y_1|^q + \frac{1}{2}|y_2|^2,$$

which is the usual total energy associated to (3.2). Clearly \hat{V} satisfies (2.5) since obviously verifies (2.2). Furthermore it is a non-increasing function of time along the solutions of (3.2). Indeed

$$\begin{aligned} \frac{d}{dt} \hat{V}(y(t)) &= \hat{V}'_{\varphi}(y(t)) = \hat{V}_y \cdot \varphi \\ &= \tau |y_1(t)|^{q-2} (y_1(t), y_2(t)) - h(t) |y_2(t)|^{\beta+2} - \tau |y_1(t)|^{q-2} (y_1(t), y_2(t)) \\ &= -h(t) |y_2(t)|^{\beta+2}, \end{aligned}$$

where \cdot and (\cdot, \cdot) are the usual inner product in \mathbb{R}^{2N} and \mathbb{R}^N , respectively, with $\|\cdot\|$ and $|\cdot|$ the corresponding norms.

Consequently

$$\frac{d}{dt} \hat{V}(y(t)) \leq 0,$$

since $h \geq 0$, and for each solution of (3.2) there is a number $\ell \geq 0$ such that

$$(3.3) \quad \lim_{t \rightarrow \infty} \hat{V}(y(t)) = \ell.$$

By the form of \hat{V} this shows that any solution of (3.2) is bounded in I (*Lagrange's stability of the system*), and also that

$$(3.4) \quad h|y_2|^{\beta+2} \in L^1(I).$$

As a second Lyapunov function we take

$$V(t, y) = k(t)(y_1, y_2),$$

where the auxiliary function k will be chosen in accordance with (2.6)–(2.8) of Theorem 1 of Section 2.

We now follow the main idea of the proof of Theorem 1. Fix a solution $y = y(t)$ of (3.2) and assume, for contradiction, that $\ell > 0$ in (3.3).

Note first that in the proof of Theorem 1 it is enough to require the validity of (2.6) along the solution $y = y(t)$ above in the following weaker form:

for every $\eta > 0$ there are functions ψ and k , satisfying (2.7) and (2.8), such that

$$(2.6)' \quad \begin{aligned} \frac{d}{dt} V(t, y(t)) &= V'_{\varphi}(t, y(t)) \leq \psi(t) - W(\eta)k(t) = \psi(t) - ck(t), \\ c &= W(\eta) > 0, \end{aligned}$$

for all t sufficiently large.

The derivative of V with respect to the system (3.2) is

$$(3.5) \quad \begin{aligned} V'_\varphi(t, y) &= V_t + V_y \cdot \varphi \\ &= k'(t)(y_1, y_2) + k(t)|y_2|^2 - k(t)h(t)|y_2|^\beta(y_1, y_2) - \tau k(t)|y_1|^q. \end{aligned}$$

Since $y = y(t)$ is bounded in I , we denote by $M > 0$ a number such that

$$|y(t)|_\infty = \max\{|y_1(t)|, |y_2(t)|\} \leq M \quad \text{in } I.$$

Consequently, by (3.5), along the solution $y = y(t)$ under consideration, we have

$$\begin{aligned} \frac{d}{dt}V(t, y(t)) &= k'(t)(y_1, y_2) - qk(t) \left[\frac{\tau}{q}|y_1|^q + \frac{1}{2}|y_2|^2 \right] + \left(1 + \frac{q}{2}\right) k(t)|y_2|^2 \\ &\quad - k(t)h(t)|y_2|^\beta(y_1, y_2) \\ &\leq M^2|k'(t)| - qk(t)\hat{V}(y) + \left(1 + \frac{q}{2}\right) k(t)|y_2|^2 - k(t)h(t)|y_2|^\beta(y_1, y_2). \end{aligned}$$

We take k , say of class $C^1(I)$, such that there is a number $C > 0$ for which

$$(3.6) \quad k(t) \leq Ch(t) \quad \text{in } I, \quad k \notin L^1(I), \quad k' \in L^1(I).$$

By (3.3) and the fact that $\hat{V}(y(\cdot))$ is non-increasing on I we have

$$\hat{V}(y(t)) \geq \ell \quad \text{for all } t \in I.$$

Fix now a number $\vartheta > 0$ which will be determined later.

If $t \in I$ and $|y_2(t)| \leq \vartheta$ then

$$\left(1 + \frac{q}{2}\right) k(t)|y_2(t)|^2 \leq \left(1 + \frac{q}{2}\right) \vartheta^2 k(t);$$

while if $t \in I$ and $|y_2(t)| > \vartheta$ then

$$\begin{aligned} \left(1 + \frac{q}{2}\right) k(t)|y_2(t)|^2 &\leq \begin{cases} \left(1 + \frac{q}{2}\right) Ch(t) \frac{|y_2(t)|^{\beta+2}}{\vartheta^\beta} & \text{if } \beta \geq 0 \\ \left(1 + \frac{q}{2}\right) Ch(t) \frac{|y_2(t)|^{\beta+2}}{M^\beta} & \text{if } -1 < \beta < 0 \end{cases} \\ &= c_1 h(t)|y_2(t)|^{\beta+2}, \end{aligned}$$

Moreover, if $t \in I$ and $|y_2(t)| \leq k(t)$ then

$$-k(t)h(t)|y_2(t)|^\beta(y_1(t), y_2(t)) \leq M[k(t)]^{\beta+2}h(t),$$

while if $t \in I$ and $|y_2(t)| > k(t)$ then

$$-k(t)h(t)|y_2(t)|^\beta(y_1(t), y_2(t)) \leq Mh(t)|y_2(t)|^{\beta+2}.$$

Hence, along the solution $y = y(t)$, it results in I

$$\begin{aligned} \frac{d}{dt}V(t, y(t)) &= M^2|k'(t)| + (c_1 + M)h(t)|y_2(t)|^{\beta+2} + M[k(t)]^{\beta+2}h(t) \\ &\quad - \ell k(t) + \left(1 + \frac{q}{2}\right) \vartheta^2 k(t). \end{aligned}$$

Now choose

$$\vartheta^2 = \frac{\ell}{2} \frac{1}{1 + q/2} = \frac{\ell}{2 + q},$$

so that

$$\frac{d}{dt}V(t, y(t)) = M^2|k'(t)| + (c_1 + M)h(t)|y_2(t)|^{\beta+2} + M[k(t)]^{\beta+2}h(t) - \frac{\ell}{2}k(t).$$

By (3.4) and the fact that $k \in C^1(I)$, $k \notin L^1(I)$ and $k' \in L^1(I)$, we have that

$$\psi(t) = M^2|k'(t)| + (c_1 + M)h(t)|y_2(t)|^{\beta+2} + M[k(t)]^{\beta+2}h(t) \in L^1_{\text{loc}}(I)$$

and in order to get (2.8) it is enough to require that

$$(3.7) \quad \lim_{t \rightarrow \infty} \frac{\int_T^t [k(s)]^{\beta+2} h(s) ds}{\int_T^t k(s) ds} = 0.$$

Hence (2.6)' holds in I with $c = \ell/2$ and ψ, k satisfying (2.7) and (2.8). Moreover

$$\liminf_{t \rightarrow \infty} V(t, y(t)) \geq -M^2 \lim_{t \rightarrow \infty} k(t) = M^2 \left[k(T) + \int_I k'(s) ds \right] > -\infty$$

since $k' \in L^1(I)$, in other words also (2.9) holds.

An application of Theorem 1 then shows that the case $\ell > 0$ in (3.3) cannot occur for the system (3.2) when k is chosen as in (3.6) and also (3.7) holds.

Now from (3.3) with $\ell = 0$ and the form of \hat{V} we conclude that

$$\lim_{t \rightarrow \infty} y(t) = 0,$$

namely the rest state $y_0 \equiv 0$ is globally asymptotically stable for the system (3.2) and consequently also the rest state $u_0 \equiv 0$ has this property for the system (3.1) when $\alpha = 0$.

We now turn to the general system (3.1) in which possibly $\alpha > 0$. We shall indicate only the points in which the argument differs essentially from the easier case $\alpha = 0$. Again we take

$$\hat{V}(y) = \frac{\tau}{q}|y_1|^q + \frac{1}{2}|y_2|^2,$$

so that now

$$\frac{d}{dt}\hat{V}(y(t)) = -h(t)|y_1(t)|^\alpha|y_2(t)|^{\beta+2}$$

along any solution $y = (y_1, y_2) = (u, u')$ of (3.1). Hence (3.3) still holds, y is bounded on I , and now

$$h|y_1|^\alpha|y_2|^{\beta+2} \in L^1(I).$$

Assume for contradiction that $\ell > 0$ in (3.3) for a fixed $y = y(t)$ as in the previous proof.

Now, as a second Lyapunov function, we take that introduced by Leoni in [16, 17], namely

$$V(t, y) = k(t)w(|y_1|)(y_1, y_2),$$

where k is the auxiliary function of the type (3.6) which will be chosen in accordance with Theorem 1, while $w : \mathbb{R}_0^+ \rightarrow [-1, 1]$ is a C^1 function such that

$$w(s) = \begin{cases} -1 & \text{if } 0 \leq s \leq \varepsilon, \\ 0 & \text{if } s = 2\varepsilon, \\ 1 & \text{if } s \geq 3\varepsilon, \end{cases}$$

where $\varepsilon > 0$ will be chosen later sufficiently small.

Thus, for all (t, y)

$$\begin{aligned} V'_\varphi(t, y) &= V_t + V_y \cdot \varphi = k'(t)w(|y_1|)(y_1, y_2) + k(t)\dot{w}(|y_1|)\left(\frac{y_1}{|y_1|}, y_2\right)(y_1, y_2) \\ &\quad + k(t)w(|y_1|)|y_2|^2 - k(t)h(t)w(|y_1|)|y_1|^\alpha|y_2|^\beta(y_1, y_2) - \tau k(t)w(|y_1|)|y_1|^q, \end{aligned}$$

and this holds also at $y_1 = 0$ since $w(s) = \text{const.}$ sufficiently close at $s = 0$, with the second term equal to zero when $y_1 = 0$.

Consequently, along the fixed solution $y = y(t)$, we have

$$(3.8) \quad \begin{aligned} V'_\varphi(t, y(t)) &= k'(t)w(|y_1|)(y_1, y_2) - k(t)h(t)w(|y_1|)|y_1|^\alpha|y_2|^\beta(y_1, y_2) \\ &\quad + k(t)w(|y_1|)[|y_2|^2 - \tau|y_1|^q] + k(t)\dot{w}(|y_1|)\left(\frac{y_1}{|y_1|}, y_2\right)(y_1, y_2). \end{aligned}$$

By using the same argument above we see that again

$$k'(t)w(|y_1(t)|)(y_1(t), y_2(t)) \leq M^2|k'(t)| \in L^1(I);$$

$$\begin{aligned}
& -k(t)h(t)w(|y_1(t)|)|y_1(t)|^\alpha|y_2(t)|^\beta(y_1(t), y_2(t)) \\
& \leq \begin{cases} M^{\alpha+1}[k(t)]^{\beta+2}h(t) & \text{if } |y_2(t)| \leq k(t) \\ Mh(t)|y_1(t)|^\alpha|y_2(t)|^{\beta+2} \in L^1(I) & \text{if } |y_2(t)| > k(t). \end{cases}
\end{aligned}$$

For the remaining two terms we apply this new argument.

Let $t \in I$ be such that $|y_1(t)| \leq 2\varepsilon$, so that $w(|y_1(t)|) \leq 0$. Hence from

$$\hat{V}(y(t)) = \frac{\tau}{q}|y_1(t)|^q + \frac{1}{2}|y_2(t)|^2 \geq \ell,$$

we get

$$|y_2(t)|^2 \geq 2\ell - \frac{2\tau}{q}(2\varepsilon)^q \geq \frac{3}{2}\ell,$$

provided that $\varepsilon > 0$ is sufficiently small. Thus

$$|y_2(t)|^2 - \tau|y_1(t)|^q \geq \frac{3}{2}\ell - \tau(2\varepsilon)^q \geq \ell,$$

with $\varepsilon > 0$ even smaller, if necessary.

Now, if $t \in I$ and moreover $|y_1(t)| \leq \varepsilon$, then $w(|y_1(t)|) = -1$ so that

$$k(t)w(|y_1(t)|)[|y_2(t)|^2 - \tau|y_1(t)|^q] \leq -\ell k(t).$$

while if $t \in I$ and $\varepsilon < |y_1(t)| \leq 2\varepsilon$, then $w(|y_1(t)|) \leq 0$ and

$$\begin{aligned}
k(t)w(|y_1(t)|)[|y_2(t)|^2 - \tau|y_1(t)|^q] & \leq 0 = -\ell k(t) + \ell k(t) \\
& \leq -\ell k(t) + \ell c h(t) \frac{|y_1(t)|^\alpha}{\varepsilon^\alpha} \frac{|y_2(t)|^{\beta+2}}{(3\ell/2)^{(\beta+2)/2}} \\
& = -\ell k(t) + c_1 h(t) |y_1(t)|^\alpha |y_2(t)|^{\beta+2},
\end{aligned}$$

and we are done.

Consider next the case $t \in I$ and $|y_1(t)| > 2\varepsilon$ for which $w(|y_1(t)|) \geq 0$. Here we distinguish the proof in two subcases either $|y_2(t)| \leq \varepsilon$ or $|y_2(t)| > \varepsilon$. If $|y_2(t)| \leq \varepsilon$ then by the form of \hat{V}

$$|y_1(t)|^q \geq \frac{q}{\tau} \left(\ell - \frac{1}{2}|y_2(t)|^2 \right) \geq \frac{q}{\tau} \left(\ell - \frac{\varepsilon^2}{2} \right) \geq (3\varepsilon)^q,$$

for $\varepsilon > 0$ possibly smaller than before. Thus, by construction, $w(|y_1(t)|) = 1$, and

$$|y_2(t)|^2 - \tau|y_1(t)|^q \leq \varepsilon^2 - q \left(\ell - \frac{\varepsilon^2}{2} \right) = \left(1 + \frac{q}{2} \right) \varepsilon^2 - q\ell \leq -\ell$$

for $\varepsilon > 0$ sufficiently small. Consequently

$$k(t)w(|y_1(t)|)[|y_2(t)|^2 - \tau|y_1(t)|^q] = k(t)[|y_2(t)|^2 - \tau|y_1(t)|^q] \leq -\ell k(t).$$

If $|y_2(t)| > \varepsilon$ then $0 \leq w(|y_1(t)|) \leq 1$, and

$$\begin{aligned} k(t)w(|y_1(t)|)[|y_2(t)|^2 - \tau|y_1(t)|^q] &\leq M^2k(t) - \ell k(t) + \ell k(t) \\ &\leq -\ell k(t) + (\ell + M^2)ch(t) \frac{|y_1(t)|^\alpha |y_2(t)|^{\beta+2}}{(2\varepsilon)^\alpha \varepsilon^{\beta+2}} \\ &= -\ell k(t) + c_2h(t)|y_1(t)|^\alpha |y_2(t)|^{\beta+2}. \end{aligned}$$

Summarizing, for all $t \in I$ we have

$$k(t)w(|y_1(t)|)[|y_2(t)|^2 - \tau|y_1(t)|^q] \leq -\ell k(t) + \max\{c_1, c_2\}h(t)|y_1(t)|^\alpha |y_2(t)|^{\beta+2}.$$

For the last term in (3.8) we first note that $w(|y_1(t)|) = 0$ if $t \in I$ is such that either $|y_1(t)| \leq \varepsilon$ or $|y_1(t)| \geq 3\varepsilon$. Hence only the case $t \in I$ and $\varepsilon < |y_1(t)| < 3\varepsilon$ is interesting. In this case, again,

$$|y_2(t)|^2 \geq 2\ell - \frac{2\tau}{q}(3\varepsilon)^q \geq \ell,$$

for $\varepsilon > 0$ sufficiently small. Thus, since w is of class $C^1(\mathbb{R}_0^+)$, the number $\max\{|\dot{w}(s)| : s \in [\varepsilon, 3\varepsilon]\} = c_3 > 0$, and

$$\begin{aligned} k(t)\dot{w}(|y_1(t)|)\left(\frac{y_1(t)}{|y_1(t)|}, y_2(t)\right)(y_1(t), y_2(t)) &\leq c_3M^3k(t) \leq c_3M^3ch(t) \frac{|y_1(t)|^\alpha |y_2(t)|^{\beta+2}}{\varepsilon^\alpha \ell^{\beta+2}} \\ &= c_4h(t)|y_1(t)|^\alpha |y_2(t)|^{\beta+2}. \end{aligned}$$

In conclusion condition (2.6)' holds provided that again (3.7) is satisfied.

As a final remark we point out that the rest state $y_0 \equiv 0$ is globally asymptotically stable for the system (3.1) if (3.6) and (3.7) hold for all $\alpha \geq 0$, namely independently from the value of the parameter α .

As a specific example we take $h(t) = a(t)t^\gamma$, with

$$\frac{1}{C} \leq a(t) \leq C \quad \text{in } I,$$

and as discussed earlier, with $\gamma \geq -1$, see Theorem 5.1 of [26]. Consequently (3.6) holds when $k(t) = 1/t$. In this case (3.7) reduces to the request that $\gamma - \beta - 1 \leq 0$. Hence we have global asymptotic stability for system (3.1) when $h(t) \cong t^\gamma$ as $t \rightarrow \infty$ whenever

$$-1 \leq \gamma \leq 1 + \beta.$$

This result is the best possible in the sense that when $\gamma < -1$ there is no stability as shown by Theorem 5.1 of [26] and when $\gamma > 1 + \beta$ in general there is no stability. In fact take

$$u'' + h(t)|u|u' + u = 0,$$

with $h(t) \cong t^\gamma$ as $t \rightarrow \infty$, $\gamma = 1 + \varepsilon$, $\varepsilon > 0$,

$$h(t) = \varepsilon^{-1}t^{1+\varepsilon} + (\varepsilon + 1)(t + t^{1-\varepsilon})^{-1}.$$

This equation admits as solution the bounded function $u(t) = 1 + t^{-\varepsilon}$ which does not approach zero as $t \rightarrow \infty$ (see also Section 1).

We end the section with the discussion of stability of the canonical classical examples presented in Section 1 using the notation and argument of Section 2.

First note that the following result holds:

Let $V = V(t, y)$ be such that

$$W_1(|y|) \leq V(t, y) \leq W_2(|y|), \quad V'_\varphi(t, y) \leq 0.$$

Then for the system $y' = \varphi(t, y)$ the rest state $y_0 \equiv 0$ is simple stable.

Now consider the celebrated Van der Pol equation

$$(3.9) \quad u'' + \varepsilon(1 - u^2)u' + u = 0, \quad \varepsilon > 0,$$

for which we shall show that $u = 0$, $u' = 0$ is an attractor.

The first Lyapunov function we choose is

$$V(y) = V(u, u') = \frac{1}{2}(u^2 + u'^2).$$

Thus

$$V'_\varphi(y) = uu' + u'u'' = -\varepsilon(1 - u^2)u'^2 \leq 0$$

in the set $B' = [-1, 1] \times \mathbb{R}$. But this function $-\varepsilon(1 - u^2)u'^2$ is not a wedge since it is zero when $u' = 0$. Hence Theorem A cannot be applied.

We shall show that the set

$$B = \{y = (u, u') \in \mathbb{R}^2 : u^2 + u'^2 < 1\}$$

is a positively invariant set for (3.9), that is if u is a solution of (3.9) which starts in B it will stay in B in future.

Indeed, let $T > 0$ be such that $u^2(T) + u'^2(T) = 1 - \vartheta < 1$, then $u^2(t) + u'^2(t) \leq 1 - \vartheta < 1$ for all $t > T$, since $B \subset B'$ and $V(u(\cdot), u'(\cdot))$ is non-increasing in $I = [T, \infty)$.

As a second Lyapunov function take

$$\hat{V}(y) = \frac{1}{2}u'^2$$

so that for all $t > T$ along the solution u

$$\begin{aligned} V'_\varphi(y(t)) &= -2\varepsilon(1 - u^2(t))\hat{V}(y(t)) \leq -2\varepsilon\vartheta\hat{V}(y(t)) = -W(\hat{V}(y(t))), \\ \hat{V}'_\varphi(y(t)) &= u'(t)u''(t) = -\varepsilon[1 - u^2(t)]u'^2(t) - u(t)u'(t) \leq |u(t)u'(t)| < 1; \end{aligned}$$

of course $V \geq 0$ and $\hat{V} \geq 0$. We can then apply Theorem 2 of Section 2 with $k \equiv 1$, $\psi \equiv 0$, $\hat{\psi} \equiv M = 1$, and conclude that

$$\lim_{t \rightarrow \infty} \hat{V}(y(t)) = 0,$$

in other words

$$\lim_{t \rightarrow \infty} u'(t) = 0.$$

Furthermore $u(t) \rightarrow 0$ as $t \rightarrow \infty$. Indeed, since $V(u(t), u'(t)) \rightarrow \ell^2/2$, say, then we have that $|u(t)| \rightarrow \ell$. Hence from (3.9) it results that also $|u''(t)| \rightarrow \ell$ and in turn $\ell = 0$ as a consequence of the Lagrange theorem.

Now rewrite (3.9) in the form

$$(3.10) \quad [u' + G(u)]' + u = 0, \quad w = u' + G(u),$$

where

$$G(u) = \int_0^u g(s)ds = \varepsilon u(1 - \frac{1}{3}u^2).$$

Thus (3.10) becomes

$$\begin{cases} u' = w - G(u) \\ w' = -u. \end{cases}$$

As a first Lyapunov function in the new variables we take

$$V(y) = V(u, w) = \frac{1}{2}(u^2 + w^2),$$

so that

$$V'_\varphi(y) = uu' + ww' = -uG(u) = -\varepsilon u^2(1 - \frac{1}{3}u^2),$$

and $\varepsilon u^2(1 - \frac{1}{3}u^2)$ is not a wedge again. Note that along any solution of (3.10) with $|u(t)| \leq \sqrt{3}$ in I , then $V(u(\cdot), w(\cdot))$ is non-increasing in I . Consequently the set

$$\hat{B} = \{y = (u, w) \in \mathbb{R}^2 : u^2 + w^2 < 3\}$$

is an invariant set for (3.10) or (3.9), namely if u is a solution of (3.10) which starts in \hat{B} then it remains in \hat{B} in future. In other words if $u^2(T) + w^2(T) = 3 - \vartheta < 3$, then $u^2(t) + w^2(t) \leq 3 - \vartheta < 3$ for all $t > T$, since $\hat{B} \subset \{y = (u, w) \in \mathbb{R}^2 : u^2 < 3\}$ and consequently $V(u(\cdot), w(\cdot))$ is non-increasing in $I = [T, \infty)$.

As a second Lyapunov function we take now

$$\hat{V}(y) = \frac{1}{2}u^2.$$

Then

$$\begin{aligned} V'_\varphi(y) &= -\frac{2}{3}\varepsilon\vartheta\hat{V}(y) = -W(\hat{V}(y)), \\ \hat{V}'_\varphi(y) &= uu' = u[w - G(u)] \leq 3 + \varepsilon\sqrt{3}. \end{aligned}$$

Hence again Theorem 2 is applicable – also the simpler Salvadori result – with $k \equiv 1$, $\psi \equiv 0$, $\hat{\psi} \equiv M = 3$. Thus $u(t) \rightarrow 0$ and as before also $w(t) \rightarrow 0$, in other words even $u'(t) \rightarrow 0$ as $t \rightarrow \infty$.

The next pictures show that for $\varepsilon \leq 1.53$ the second method which uses the form (3.10) gives a better result, while for $\varepsilon > 1.53$ the two methods give overlapping results. Moreover, for very large ε the first method which uses the form (3.9) gives a preferable stability region.

4. Abstract Evolution Equations

In this section we shall present a new asymptotic stability theorem for abstract evolution equations, which extends the earlier work, in the same direction, due to Marcati [22, 23] and Nakao [25].

We consider the following equation

$$(4.1) \quad [P(u'(t))]' + A(u(t)) + Q(t, u'(t)) + F(u(t)) = 0,$$

where $t \in I = [T, \infty)$, $T \geq 0$, and A , F , P , Q are nonlinear operators defined on appropriate Banach spaces. We understand P to be an evolution operator, A a differential operator of divergence form, Q a damping term and F a restoring force.

Concrete examples of (4.1) include the principal case of wave systems where $P = I$, $A = -\Delta$, or more generally $A = -\Delta_p$, $p > 1$, the p -Laplacian operator, or $A = (-\Delta)^L$, $L \in \mathbb{N}$, the polyharmonic operator. Also the parabolic case is included in (4.1) when, for instance, $P = 0$, $A = -\Delta$ and $Q(t, v) = |v|^{m-2}B(t)v$, $m > 1$, where B is a continuous matrix such that $(B(t)v, v) > 0$ for $v \neq 0$. Of course (4.1) includes also second order ordinary differential systems when $A = 0$.

The other main purpose of this section is to formulate a careful definition of (4.1) which is suitably general for existence purposes, independent of detailed properties of F and Q , and adequate for the investigation of asymptotic stability.

We assume that

$$A : W \rightarrow W', \quad F : X \rightarrow X', \quad P : V \rightarrow V',$$

where W, X, V are real Banach spaces, with W', X', V' their dual spaces, which admit a common continuously embedded subspace $U \neq \{0\}$.

This assumption is usually verified in concrete examples when W, X, V are the standard Lebesgue or Sobolev spaces. In this case, for some embedding theorem, they are the subspaces of a common vector space Z , so that we can take $U = W \cap X \cap V$. For instance, for the special wave system

$$u_{tt} - \Delta u + |u_t|^{m-2} B(t) u_t + |u|^{p-2} M(x) u = 0,$$

where B, M are continuous matrices in $t \in I$ and $x \in \Omega = \Omega^0 \subset \mathbb{R}^n$, Ω bounded, respectively, then we can take

$$W = [H_0^1(\Omega)]^N, \quad V = [L^2(\Omega)]^N, \quad X = [L^p(\Omega)]^N,$$

with, say, $2 \leq m \leq p$. In this case

$$X, W \subset V \quad \text{and} \quad U = X \cap W.$$

Note that $W \subset X$ when $p \leq 2^* = 2n/(n-2)$, where 2^* is the critical Sobolev exponent. Hence here $U = W$.

For ordinary differential systems $A = 0$ and $U = X = V = \mathbb{R}^N$, endowed with the usual norm.

We now list the structural conditions on the system (4.1) and comment only the ones that need some explanations.

(S1) A, F and P are Fréchet derivatives (F-derivatives) of C^1 potentials

$$\mathcal{A} : W \rightarrow \mathbb{R}, \quad \mathcal{F} : X \rightarrow \mathbb{R}, \quad \mathcal{P} : V \rightarrow \mathbb{R},$$

respectively, where, without loss of generality, we assume $\mathcal{A}(0) = 0$, $\mathcal{F}(0) = 0$ and $\mathcal{P}(0) = 0$;

(S2) $\mathcal{A}(u) + \mathcal{F}(u) \geq 0$ in U .

Note that (S2) is automatic whenever A is a positive operator, e.g. $-\Delta$, and F is a restoring force. Indeed, in this case,

$$\langle A(u), u \rangle_W \geq 0 \quad \text{in } W, \quad \langle F(u), u \rangle_X > 0 \quad \text{for } u \neq 0, u \in X,$$

and in turn

$$\mathcal{A}(u) = \int_0^1 \langle A(su), u \rangle_W ds \geq 0, \quad \mathcal{F}(u) = \int_0^1 \langle F(su), u \rangle_X ds \geq 0.$$

(S3) $\mathcal{P}^*(v) := \langle P(v), v \rangle_V - \mathcal{P}(v) \geq 0$ in V .

This condition says that the Legendre transform of the operator P is non-negative, which is always the case when P is a monotone operator, or, equivalently, when \mathcal{P} is a convex functional.

Remarks. 1. If $P = I$, then V must be a Hilbert space since $P : V \rightarrow V'$ and $V' \cong V$ by the Riesz theorem. In this case

$$\langle P(v), \phi \rangle_V = \langle v, \phi \rangle_V \quad \text{for all } v, \phi \in V.$$

Furthermore

$$\mathcal{P}(v) = \frac{1}{2} \|v\|_V^2 \quad \text{and} \quad \mathcal{P}^*(v) = \frac{1}{2} \|v\|_V^2.$$

If $A(u) = -\Delta u$, then $W = [H_0^1(\Omega)]^N$, as noted above, so W is a Hilbert space and denoting by Du the Jacobian $N \times n$ matrix of u , then

$$\langle A(u), \phi \rangle_W = \int_{\Omega} (Du, D\phi) dx,$$

where, of course, (\cdot, \cdot) is here the Euclidean inner product on the space of the $N \times n$ matrices. One finds again that

$$\mathcal{A}(u) = \frac{1}{2} \|Du\|_2^2 = \frac{1}{2} \|u\|_W^2,$$

where $\|\cdot\|_2 = \|\cdot\|_{[L^2(\Omega)]^{N \times n}}$. Combining both results we find formally

$$[P(u')] + A(u) = u_{tt} - \Delta u, \quad \text{whenever } u \in C(I \rightarrow W) \cap C^1(I \rightarrow [L^2(\Omega)]^N).$$

2. Another interesting case is given by

$$P(v) = |v|^{\nu-2}v, \quad V = [L^\nu(\Omega)]^N, \quad \nu > 1,$$

then

$$V' = [L^{\nu'}(\Omega)]^N, \quad \text{with } \frac{1}{\nu} + \frac{1}{\nu'} = 1.$$

Moreover

$$\langle P(v), \phi \rangle_V = \int_{\Omega} |v|^{\nu-2}(v, \phi) dx,$$

and

$$\mathcal{P}(v) = \frac{1}{\nu} \|v\|_V^\nu, \quad \mathcal{P}^*(v) = \frac{1}{\nu'} \|v\|_V^\nu,$$

where as already noted $V = [L^\nu(\Omega)]^N$.

One can do the same with F . For example, take

$$F(u) = |u|^{q-2}u \quad \text{and} \quad X = [L^q(\Omega)]^N, \quad q > 1,$$

and so forth.

3. Note that if $P = 0$, we obtain the parabolic case thanks to the presence of the time t in the nonlinear damping Q of (4.1).

4. If $A = 0$, then (4.1) reduces to an ordinary differential equation in Banach spaces.

We introduce now the further new key idea which allow a great generality in (4.1). Let S be a subset of $I \times V$ and

$$Q : S \rightarrow U'$$

be continuous. Let us denote by K the set of all function $u : I \rightarrow U$ such that

$$u \in C^1(I \rightarrow V) \cap C(I \rightarrow U).$$

Definition. We say that u is a *strong solution* of the system (4.1) if

- (a) $u \in K$ and $(t, u'(t)) \in S$ for a.a. $t \in I$;
- (b) u verifies (4.1) in the following distributional sense

$$\begin{aligned} \langle P(u'(s)), \phi(s) \rangle_V \Big|_T^t &= \int_T^t \{ \langle P(u'(s)), \phi'(s) \rangle_V - \langle A(u(s)), \phi(s) \rangle_W \\ &\quad - \langle Q(s, u'(s)), \phi(s) \rangle_U - \langle F(u(s)), \phi(s) \rangle_X \} ds \end{aligned}$$

for all $\phi \in K$ and $t \in I$;

- (c) there is a measurable function $\mathcal{D}u : I \rightarrow \mathbb{R}_0^+$ such that

$$\mathcal{E}u(t) + \int_T^t \mathcal{D}u(s) ds$$

is non-increasing on I – *weak conservation condition* – where $\mathcal{E}u : I \rightarrow \mathbb{R}_0^+$ is the *total energy function associated to u* , given by

$$(4.2) \quad \mathcal{E}u(t) = \mathcal{P}^*(u'(t)) + \mathcal{A}(u(t)) + \mathcal{F}(u(t)).$$

For the expression

$$\langle [P(u'(s))]', \phi(s) \rangle_V$$

we have to use a definition which fits with the fact that P usually is not assumed to be differentiable. We define

$$\langle [P(u'(s))]', \phi(s) \rangle_V := \langle P(u'(s)), \phi(s) \rangle_V' - \langle P(u'(s)), \phi'(s) \rangle_V$$

for all $u, \phi \in K$. Consequently we obtain the first two terms in (b) by integration.

We shall show that all the pairings in the distribution identity (b) are well defined along their time integrals, except possibly the fourth pairing which will become meaningful too with the help of the further structural condition (S4) which will be given later.

Rewrite (b) as

$$I_1 = \int_T^t \{I_2 + I_3 + I_4 + I_5\} ds.$$

Of course I_1 is well defined by definition of K . Also I_2 is well defined, since $P(u') : I \rightarrow V'$ and $\phi' : I \rightarrow V$ are continuous and in turn

$$\langle P(u'(\cdot)), \phi(\cdot) \rangle_V : I \rightarrow \mathbb{R}$$

is continuous and so in $L^1_{\text{loc}}(I)$ as needed in (b). For I_3 we can repeat the argument above, namely we again get that

$$\langle A(u(\cdot)), \phi(\cdot) \rangle_W : I \rightarrow \mathbb{R}$$

is continuous and so in $L^1_{\text{loc}}(I)$. For I_4 we can only say up to now that is defined a.e. in I by (a) and is measurable on I by the continuity of Q and the choice of K . We shall later add the other structural condition (S4) on Q which will yield

$$I_4 \in L^1_{\text{loc}}(I).$$

For I_5 we see again that

$$\langle F(u(\cdot)), \phi(\cdot) \rangle_X \in C(I) \subset L^1_{\text{loc}}(I),$$

and we are done.

Of course, the solutions in our definition are (in general) not classical in any sense and the word *strong* is to emphasize that the test space for the functions ϕ is precisely the set K in which solutions reside, while in contrast the typical test space for distributional (*weak*) solutions is usually a smaller functional space. The choice of the test space K is essential in the definition of our second Lyapunov function to prove asymptotic stability for the system (4.1).

Note that by (S1)–(S3) and the definition of K the function $\mathcal{E}u$ maps I into \mathbb{R}_0^+ continuously.

The definition (4.2) arises from the (formal) requirement:

$$(\mathcal{E}u)'(t) = \langle [P(u'(t))]', u(t) \rangle_V + \langle A(u(t)), u'(t) \rangle_W + \langle F(u(t)), u'(t) \rangle_X.$$

But (formally)

$$[\mathcal{A}(u(t))]' = \langle A(u(t)), u'(t) \rangle_W.$$

This is not rigorous since it is known only that $u' : I \rightarrow V$, while we need here $u'(t) \in W$. Also (formally)

$$[\mathcal{F}(u(t))]' = \langle F(u(t)), u'(t) \rangle_X.$$

so that here we need $u'(t) \in X$. Lastly, again formally,

$$\begin{aligned} [\mathcal{P}^*(u'(t))]' &= [\langle P(u'(t)), u'(t) \rangle_V - \mathcal{P}(u'(t))]' = \langle P(u'(t)), u'(t) \rangle'_V - \langle P(u'(t)), u'(t) \rangle_V \\ &= \langle [P(u'(t))]', u'(t) \rangle_V \end{aligned}$$

as already noted above. Thus $\mathcal{E}u$ is an appropriate *energy* for (4.1).

Since $\mathcal{D}u \geq 0$, the integral $\int_T^t \mathcal{D}u(s)ds$ is well defined, possibly ∞ . But then from (c) and the fact that $\mathcal{E}u \geq 0$ in I we see that necessarily $\int_T^t \mathcal{D}u(s)ds$ is *bounded for all $t \in I$* , namely is bounded on I . Hence

$$\mathcal{D}u \in L^1(I).$$

In turn $t \mapsto \int_T^t \mathcal{D}u(s)ds$ is non-decreasing, yielding finally that $\mathcal{E}u$ is non-increasing on I .

Note that if

$$\mathcal{E}u(t) + \int_T^t \mathcal{D}u(s)ds \equiv \text{Const.} \quad \text{in } I,$$

then we would have $\mathcal{E}u \in AC(I)$ and

$$(\mathcal{E}u)'(t) = -\mathcal{D}u(t) \leq 0 \quad \text{a.e. in } I,$$

which is the standard energy dissipation identity. In fact for a first Lyapunov function we need only

$$(\mathcal{E}u)'(t) = -\mathcal{D}u(t) \leq 0 \quad \text{a.e. in } I,$$

this is the reason for the requirement that

$$\mathcal{E}u(t) + \int_T^t \mathcal{D}u(s)ds$$

is *non-increasing* on I .

In the case of ordinary differential systems the relation between $\mathcal{D}u$ and Q is very simple being

$$\mathcal{D}u(t) = (Q(t, u'(t)), u'(t)), \quad t \in I.$$

As an example consider

$$u'' + Q(t, u') + u = 0,$$

where $Q : I \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a continuous damping function as in Section 2. Then the energy

$$\mathcal{E}u(t) = V(y) = \frac{1}{2}|u|^2 + \frac{1}{2}|u'|^2$$

and

$$(\mathcal{E}u)'(t) = V_\varphi'(y) = -(Q(t, u'(t)), u'(t)) = -\mathcal{D}u(t), \quad t \in I.$$

Consequently energy is conserved on solutions, namely by integration

$$\mathcal{E}u(t) + \int_T^t \mathcal{D}u(s)ds = V(y(t)) + \int_T^t (Q(s, u'(s)), u'(s))ds \equiv \text{Const.} \quad \text{in } I.$$

Hence (c) is the *weak version* of energy conservation for (4.1) which extends the standard case above to abstract equations.

We now postulate the following crucial connection condition between *the rate of dissipation of energy* $\mathcal{D}u$ of any strong solution $u = u(t)$ of (4.1) and Q .

(S4) there are an exponent $m > 1$ and a non-negative function $\delta \in L^1_{\text{loc}}(I)$ such that along any solution $u = u(t)$ of (4.1)

$$(4.3) \quad \|Q(t, u'(t))\|_{U'} \leq \delta(t)^{1/m} \mathcal{D}u(t)^{1/m'} \quad \text{a.e. in } I,$$

where as usual m' is the Hölder conjugate of m , and there are a non-negative function σ with $1/\sigma \in L^{m-1}_{\text{loc}}(I)$ and a wedge W such that along any solution $u = u(t)$ of (4.1)

$$(4.4) \quad \mathcal{D}u(t) \geq \sigma(t)W(\|u'(t)\|_V) \quad \text{a.e. in } I.$$

Note that in the canonical second order ordinary differential case in which $P = I$, $A = 0$, $V = X = U = \mathbb{R}^N$, and, e.g.,

$$Q(t, v) = h(t)|v|^\beta v, \quad h \in C(I \rightarrow \mathbb{R}_0^+), \quad \beta > -1,$$

along any solution we have

$$\mathcal{D}u(t) = (Q(t, u'(t)), u'(t)) = h(t)|u'(t)|^{\beta+2}.$$

Hence we need

$$h(t)|u'(t)|^{\beta+1} = \|Q(t, u'(t))\|_{U'} \leq \delta^{1/m} h(t)^{1/m'} |u'(t)|^{(\beta+2)/m'}.$$

which holds provided that

$$m = \beta + 2 \quad \text{and} \quad \delta = h.$$

Also the second growth condition (4.4) in (S4) holds, with

$$\sigma = h \quad \text{and} \quad W(\tau) = \tau^m, \quad \tau \geq 0.$$

In the distributional identity (b) the term

$$I_4 = \int_T^t \langle Q(s, u'(s)), \phi(s) \rangle_U ds$$

must still be shown to be meaningful. Note first that by (4.3) and Hölder's inequality

$$\begin{aligned} \int_T^t \|Q(s, u'(s))\|_{U'} ds &\leq \int_T^t \delta(s)^{1/m} \mathcal{D}u(s)^{1/m'} ds \leq \left(\int_T^t \delta(s) ds \right)^{1/m} \left(\int_T^t \mathcal{D}u(s) ds \right)^{1/m'} \\ &\leq C(t), \end{aligned}$$

since $\delta \in L^1_{\text{loc}}(I)$ by (S4) and $\mathcal{D}u \in L^1(I)$. Hence

$$\|Q(\cdot, u'(\cdot))\|_{U'} \in L^1_{\text{loc}}(I),$$

and so also

$$|\langle Q(t, u'(t)), \phi(t) \rangle_U| \leq \|Q(t, u'(t))\|_{U'} \cdot \|\phi(t)\|_U \in L^1_{\text{loc}}(I),$$

because $\|\phi(\cdot)\|_U \in L^\infty_{\text{loc}}(I)$ being $\phi \in K$. Consequently all the terms in (b) are meaningful.

Note that in general (b) does not imply that $t \mapsto \mathcal{E}u(t) + \int_T^t \mathcal{D}u(s) ds$ is non-increasing on I , while this implication holds in the ordinary differential subcase of (4.1).

Finally, we assume the last two structural hypotheses.

(S5) For all $d > 0$ the sets

$$E = \{v \in V : \mathcal{P}^*(v) \leq d\} \quad \text{and} \quad P(E)$$

are bounded in V and V' , respectively.

(S6) For every $\ell > 0$ there exists $\alpha = \alpha(\ell) > 0$ such that $u \in U$ and $\mathcal{A}(u) + \mathcal{F}(u) \geq \ell$ implies that $\langle A(u), u \rangle_W + \langle F(u), u \rangle_X \geq \alpha$.

When $P(v) = |v|^{\nu-2}v$, $V = [L^\nu(\Omega)]^N$, $\nu > 1$, then for all $e \in E$

$$\|e\|_V \leq (d\nu')^{1/\nu} \quad \text{and} \quad \|P(e)\|_{V'} \leq (d\nu')^{1/\nu'},$$

namely (S5) holds.

Thus (S5) is trivially true also in the canonical second order ordinary differential case in which $\nu = 2$, or $P = I$. Moreover also (S6) holds whenever the force F is restoring, namely $\langle F(u), u \rangle > 0$ for $u \neq 0$, $u \in \mathbb{R}^N$, since $A = 0$ in this case.

For the wave systems where $A = -\Delta$, and, e.g., $F(u) = u$, then (S6) holds with $\alpha(\ell) = 2\ell$.

We now state our main stability result for the abstract evolution equation (4.1).

Theorem 4.1. *Assume that (S1)–(S6) hold. Suppose also that there is an auxiliary function $k : I \rightarrow \mathbb{R}_0^+$, $k \neq 0$, of class $C^1(I)$, such that*

$$(4.5) \quad \lim_{t \rightarrow \infty} \frac{\int_T^t |k'(s)| ds}{\int_T^t k(s) ds} = 0,$$

$$(4.6) \quad \liminf_{t \rightarrow \infty} \frac{\int_T^t [\delta(s) + \sigma^{1-m}] k(s)^m ds}{\int_T^t k(s) ds} = M < \infty.$$

Then for every $\|\cdot\|_U$ -bounded strong solution of (4.1) it results

$$(4.7) \quad \lim_{t \rightarrow \infty} \mathcal{E}u(t) = 0;$$

in other words $u_0 \equiv 0$ is a global attractor for (4.1).

From (4.5) it follows immediately that $k \notin L^1(I)$, since $k \neq 0$.

Suppose that $Q(t, v) = h(t)|v|^{m-2}v$, where $h \geq 0$, is a continuous function on I , and $m > 1$, as in the canonical ordinary differential case. Assume also that $k(t) \leq Ch(t)$ in I for some $C > 0$, as required in the simpler case (3.6). Since here (S4) holds for $\sigma = \delta = h$, condition (4.6) reduces to

$$\liminf_{t \rightarrow \infty} \frac{\int_T^t h(s)k(s)^m ds}{\int_T^t k(s) ds} < \infty.$$

In general, if $k \equiv 1$ condition (4.6) becomes

$$\liminf_{t \rightarrow \infty} \frac{1}{t^m} \int_T^t [\delta(s) + \sigma^{1-m}] ds < \infty.$$

This generalizes a result of Nakao [25], for which $\sigma = \delta \geq \text{Const.} > 0$, and

$$\int_T^t \delta(s) ds \leq \text{Const.} t^m.$$

In [25] furthermore $\mathcal{D}u(t) = \langle Q(t, u'(t)), u'(t) \rangle_Y$, where Y is a Banach space such that $X \subset Y \subset V$ continuously, $S = I \times Y$, V is a Hilbert space, $P = I$, and (a) and (S4) are required in a stronger form.

If also $\sigma(t) \geq \text{Const.}/t$ then (4.6) takes the form

$$\liminf_{t \rightarrow \infty} \frac{1}{t^m} \int_T^t \delta(s) ds < \infty.$$

Before proving Theorem 4.1 we give the following

Lemma. Assume (S1)–(S6). Then along any strong solution u of (4.1)

(i) $\mathcal{E}u$ is non-increasing in I with

$$0 \leq \mathcal{E}u(t) \leq \mathcal{E}u(T) \quad \text{and} \quad \mathcal{P}^*(u'(t)) \leq \mathcal{E}u(T) \quad \text{in } I;$$

(ii) $\mathcal{D}u \in L^1(I)$.

(iii) $\langle A(u), u \rangle_W + \langle F(u), u \rangle_X \geq 0$ in U .

Proof. Properties (i) and (ii) are direct consequences of (c) and (S2), as already noted when (c) was introduced. To prove (iii) assume for contradiction that there is a point $w \in U$ such that

$$(4.8) \quad \langle A(w), w \rangle_W + \langle F(w), w \rangle_X < 0.$$

By (S2) we know that $\mathcal{A}(w) + \mathcal{F}(w) = \ell \geq 0$. Now, if $\ell > 0$ then by (S6) there is $\alpha = \alpha(\ell) > 0$ such that

$$\langle A(w), w \rangle_W + \langle F(w), w \rangle_X \geq \alpha > 0,$$

which contradicts (4.8).

On the other hand, if $\ell = 0$ then by (S2) the point w is a minimum for the C^1 functional $\mathcal{A} + \mathcal{F}$, in other words the function

$$[1 - \varepsilon, 1 + \varepsilon] \ni \tau \mapsto (\mathcal{A} + \mathcal{F})(\tau w) \in \mathbb{R}$$

has a minimum at $\tau = 1$. Consequently the F-derivative $A + F$ must be zero at w , which contradicts (4.8) again.

This completes the proof of (iii) and of the lemma.

Proof of Theorem 4.1. Suppose for contradiction that the assertion (4.7) fails along some $\|\cdot\|_U$ -bounded strong solution u of (4.1). Put $\sup\{\|u(t)\|_U : t \in I\} = B$. Then by Lemma (i) above there exists $\ell > 0$ such that

$$(4.9) \quad \mathcal{E}u(t) \searrow 2\ell \quad \text{as } t \rightarrow \infty.$$

Since $k \in C^1(I)$ the function $\phi = ku$ is in K and we can evaluate (b) along such ϕ . Hence for all $t \geq R \geq T$ we have

$$\begin{aligned} G(s)\Big|_R^t &:= k(s)\langle P(u'(s)), u(s) \rangle_V \Big|_R^t = \int_R^t \{k'(s)\langle P(u'(s)), u(s) \rangle_V \\ &\quad + k(s)[\langle P(u'(s)), u'(s) \rangle_V + \mathcal{P}^*(u'(s))] - k(s)\langle Q(s, u'(s)), u(s) \rangle_U \\ &\quad - k(s)[\mathcal{P}^*(u'(s)) + \langle A(u(s)), \phi(s) \rangle_W + \langle F(u(s)), \phi(s) \rangle_X]\} ds \\ &:= \int_R^t \{J_1 + J_2 + J_3 + J_4\} ds. \end{aligned}$$

By Lemma (i) and (S5) there is a number $C > 0$ such that

$$\sup\{\|u'(t)\|_V : t \in I\}, \sup\{\|P(u'(t))\|_{V'} : t \in I\} \leq C.$$

Consequently

$$\begin{aligned} \int_R^t J_1 ds &\leq \int_R^t |k'(s)| \cdot \|P(u'(s))\|_{V'} \cdot \|u(s)\|_V ds \leq Cb \int_R^t \|u(s)\|_U \cdot |k'(s)| ds \\ &\leq bBC \int_R^t |k'(s)| ds, \end{aligned}$$

since $U \subset V$ continuously so that $\|u\|_V \leq b\|u\|_U$ for all $u \in U$ and some $b > 0$.

Next, note that

$$J_2 \leq [C^2 + \mathcal{E}u(T)]k \quad \text{in } I$$

by Lemma (i) and the choice of C . Now fix $\vartheta > 0$. By (S1) and (S3) the functions P and \mathcal{P}^* are continuous on V , with $\mathcal{P}^*(0) = 0$. Hence there exists $\Lambda(\vartheta) > 0$ such that

$$\langle P(v), v \rangle_V + \mathcal{P}^*(v) \leq \vartheta \quad \text{whenever } v \in V \text{ and } \|v\|_V \leq \Lambda(\vartheta).$$

Therefore

$$J_2 \leq k(t) \begin{cases} \vartheta & \text{if } t \in I \text{ and } \|u'(t)\|_V \leq \Lambda(\vartheta) \\ [C^2 + \mathcal{E}u(T)] & \text{if } t \in I \text{ and } \|u'(t)\|_V > \Lambda(\vartheta). \end{cases}$$

For simplicity denote by J the first set of $t \in I$. Now in $I \setminus J$ by (S4) we have

$$\frac{\mathcal{D}u(t)}{\sigma(t)} \geq W(\|u'(t)\|_V) \geq W(\Lambda(\vartheta)) > 0 \quad \text{a.e. in } I \setminus J.$$

Thus

$$\begin{aligned} J_2 &\leq k(t) \begin{cases} \vartheta & \text{in } J \\ [C^2 + \mathcal{E}u(T)] \left[\frac{\mathcal{D}u(t)}{W(\Lambda(\vartheta))\sigma(t)} \right]^{1/m'} & \text{a.e in } I \setminus J \end{cases} \\ &= \begin{cases} \vartheta k(t) & \text{in } J \\ C_1 [\sigma(t)^{1-m} k(t)^m]^{1/m} \mathcal{D}u(t)^{1/m'} & \text{a.e in } I \setminus J, \end{cases} \end{aligned}$$

where

$$C_1(\vartheta) = [C^2 + \mathcal{E}u(T)]W(\Lambda(\vartheta))^{-1/m'}.$$

Consequently by Hölder's inequality

$$\begin{aligned} \int_R^t J_2 ds &\leq \vartheta \int_R^t k(s) ds + C_1(\vartheta) \left(\int_R^t \sigma(s)^{1-m} k(s)^m ds \right)^{1/m} \left(\int_R^t \mathcal{D}u(s) ds \right)^{1/m'} \\ &\leq \vartheta \int_R^t k(s) ds + C_1(\vartheta) \varepsilon(R) \left(\int_R^t \sigma(s)^{1-m} k(s)^m ds \right)^{1/m}, \end{aligned}$$

where

$$\varepsilon(R) = \left(\int_R^\infty \mathcal{D}u(s) ds \right)^{1/m'} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

since $\mathcal{D}u \in L^1(I)$ by Lemma (ii).

Now by (S4) and Hölder's inequality

$$\begin{aligned} \int_R^t J_3 ds &\leq \int_R^t k(s) \|Q(s, u'(s))\|_{U'} \|u(s)\|_U ds \leq B \int_R^t k(s) \delta(s)^{1/m} \mathcal{D}u(s)^{1/m'} ds \\ &\leq B \left(\int_R^t \delta(s) k(s)^m ds \right)^{1/m} \left(\int_R^t \mathcal{D}u(s) ds \right)^{1/m'} \leq B\varepsilon(R) \left(\int_T^t \delta(s) k(s)^m ds \right)^{1/m}. \end{aligned}$$

Next, if $\mathcal{P}^*(u'(t)) \geq \ell$ at some $t \in I$, then by (iii) we have $J_4(t) \leq -\ell k(t)$. On the other hand, if $\mathcal{P}^*(u'(t)) < \ell$, then by Lemma (i) and (4.9)

$$\mathcal{A}(u(t)) + \mathcal{F}(u(t)) = \mathcal{E}u(t) - \mathcal{P}^*(u'(t)) > 2\ell - \ell = \ell.$$

Hence, by (S6) and the fact that $u(t) \in U$, we get

$$\langle \mathcal{A}(u(t)), u(t) \rangle_W + \langle \mathcal{F}(u(t)), u(t) \rangle_X \geq \alpha(\ell),$$

and, without loss of generality, we can take $\alpha(\ell) \leq \ell$. Since $\mathcal{P}^*(u'(t)) \geq 0$ by (S3) we obtain $J_3(t) \leq -\alpha(\ell)k(t)$ when $\mathcal{P}^*(u'(t)) \leq \ell$. Consequently

$$J_3(t) \leq -\alpha(\ell)k(t) \quad \text{in } I.$$

Combining all the previous steps yields for all $t \geq R \geq T$

$$\begin{aligned} G(s)|_R^t &\leq bBC \int_R^t |k'(s)| ds + \vartheta \int_R^t k(s) ds + C_1(\vartheta)\varepsilon(R) \left(\int_R^t \sigma(s)^{1-m} k(s)^m ds \right)^{1/m} \\ &\quad + B\varepsilon(R) \left(\int_R^t \delta(s) k(s)^m ds \right)^{1/m} - \alpha(\ell) \int_R^t k(s) ds. \end{aligned}$$

Since $G(t) = k(t) \langle P(u'(t)), u(t) \rangle_V$, as shown above, we get in I

$$|G(t)| \leq bBCk(t) \leq bBC \left\{ k(T) + \int_T^t |k'(s)| ds \right\}.$$

Consequently

$$\begin{aligned} (4.10) \quad -G(R) &\leq bBCk(T) + 2bBC \int_T^t |k'(s)| ds + \vartheta \int_R^t k(s) ds \\ &\quad + C_1(\vartheta)\varepsilon(R) \left(\int_T^t \sigma(s)^{1-m} k(s)^m ds \right)^{1/m} \\ &\quad + B\varepsilon(R) \left(\int_T^t \delta(s) k(s)^m ds \right)^{1/m} - \alpha(\ell) \int_R^t k(s) ds. \end{aligned}$$

Now for $t \geq R$

$$\int_T^t |k'(s)| ds \leq \varepsilon_1(R) \int_T^t k(s) ds,$$

where $\varepsilon_1(R) \rightarrow 0$ as $R \rightarrow \infty$ by (4.5). Moreover, by (4.6) there is a sequence $t_j \nearrow \infty$ such that for all j

$$\int_T^{t_j} [\delta(s) + \sigma(s)^{1-m}] k(s)^m ds \leq (M+1) \left(\int_T^{t_j} k(s) ds \right)^m.$$

Thus for all $R \geq T$ and all t_j with $t_j \geq R$ we have by (4.10)

$$(4.11) \quad \begin{aligned} -G(R) &\leq bBCk(T) + \{2bBC\varepsilon_1(R) + \vartheta \\ &\quad + (M+1)^{1/m}\varepsilon(R)[C_1(\vartheta) + B]\} \int_T^{t_j} k(s) ds \\ &\quad - \alpha(\ell) \int_T^{t_j} k(s) ds + \alpha(\ell) \int_T^R k(s) ds. \end{aligned}$$

Now choose $\vartheta > 0$ such that

$$\vartheta \leq \frac{1}{3}\alpha(\ell)$$

and next choose R so large that

$$2bBC\varepsilon_1(R) + \vartheta + (M+1)^{1/m}\varepsilon(R)[C_1(\vartheta) + B] \leq \frac{2}{3}\alpha(\ell).$$

Consequently by (4.11) we have

$$\begin{aligned} -\text{Const.} &:= -G(R) - bBCk(T) - \alpha(\ell) \int_T^R k(s) ds \\ &\leq \left(\frac{2}{3} - 1 \right) \alpha(\ell) \int_T^{t_j} k(s) ds \longrightarrow -\infty \end{aligned}$$

as $j \rightarrow \infty$, since $k \notin L^1(I)$. This is obviously impossible and completes the proof of the theorem.

The special case of wave systems

Consider now the concrete example given by the system

$$(4.12) \quad \begin{cases} u_{tt} - \Delta u + Q(t, x, u_t) + f(x, u) = 0 & \text{in } I \times \Omega, \\ u(t, x) = 0 & \text{on } I \times \partial\Omega, \end{cases}$$

where $t \in I = [T, \infty)$, $x \in \Omega = \Omega^0$, Ω bounded domain of \mathbb{R}^n , $n \geq 2$, say.

Assume $f \in C(I \times \mathbb{R}^N \rightarrow \mathbb{R}^N)$, $Q \in C(I \times \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N)$, $(f(x, u), u) \geq 0$, $(Q(t, x, v), v) \geq 0$, with $f = D_u \Phi$ and $\Phi(x, 0) \equiv 0$ in Ω , such that

$$(4.13) \quad \begin{aligned} |f(x, u)| &\leq \text{Const.} (1 + |u|^{p-1}), \quad p > 1, \\ |Q(t, x, v)| &\leq \delta(t)^{1/m} (Q(t, x, v), v)^{1/m'}, \quad m > 1, \quad \frac{1}{m} + \frac{1}{m'} = 1, \end{aligned}$$

and $\delta \in L^1_{\text{loc}}(I \rightarrow \mathbb{R}_0^+)$.

We have to find the spaces V , W , X and the operators \mathcal{P} , \mathcal{A} , \mathcal{F} so that the system (4.12) can be viewed as a special case of (4.1). As already shown at the beginning of this section, here $P = I$ with $V = [L^2(\Omega)]^N$ and $A = -\Delta$ with $W = [H_0^1(\Omega)]^N$. Note that by (4.13)₁

$$0 \leq \Phi(x, u) = \int_0^1 (f(x, su), u) ds \leq \text{Const.} (|u| + |u|^p).$$

Thus

$$\mathcal{F}(u(t)) = \int_{\Omega} \Phi(x, u(t, x)) dx \leq \text{Const.} (\|u(t, \cdot)\|_p + \|u(t, \cdot)\|_p^p)$$

and so $X = [L^p(\Omega)]^N$. Moreover

$$\int_{\Omega} |f(x, u(t, x))|^{p'} dx \leq \text{Const.} (1 + \|u(t, \cdot)\|_p^p),$$

hence

$$F(u(t)) := f(\cdot, u(t, \cdot)) : X \rightarrow X',$$

since p' is the Hölder conjugate exponent of p . In this case, of course,

$$U = V \cap W \cap X,$$

since V , W , X are at this moment subspaces of the same space $Z = [L^1(\Omega)]^N$.

It remains to choose S and $\mathcal{D}u$. Define

$$S = \{(t, v) \in I \times V : \|Q(t, \cdot, v)\|_{p'} < \infty\},$$

and

$$\mathcal{D}u(t) = \int_{\Omega} (Q(t, x, u_t(t, x), u_t(t, x))) dx.$$

By this choice

$$Q : S \rightarrow X' \subset U',$$

and is continuous. We have to show that

$$(t, u_t(t, \cdot)) \in S \quad \text{for a.a. } t \in I.$$

To see this, first recall that here

$$\mathcal{E}u(t) = \frac{1}{2}\|u_t(t, \cdot)\|_2^2 + \frac{1}{2}\|Du(t, \cdot)\|_2^2 + \mathcal{F}(u(t))$$

and

$$\mathcal{E}u(t) + \int_T^t \mathcal{D}u(s)ds = \mathcal{E}u(t) + \int_T^t \int_{\Omega} (Q(s, x, u_t(s, x), u_t(s, x)))dxds$$

which is non-increasing on I , so that

$$\mathcal{D}u(t) = \int_{\Omega} (Q(t, x, u_t(t, x), u_t(t, x)))dx \in L^1(I).$$

The condition $(t, u_t(t, \cdot)) \in S$ a.e. in I is the same as

$$\|Q(t, \cdot, u_t(t, \cdot))\|_{p'} < \infty \quad \text{a.e. in } I.$$

We take $m \leq p$ so that $p' \leq m'$ and it is enough to show that

$$\int_{\Omega} |Q(t, x, u_t(t, x))|^{m'} dx < \infty \quad \text{a.e. in } I.$$

Indeed, by the principal condition (4.13)₂ we have

$$\int_{\Omega} |Q(t, x, u_t(t, x))|^{m'} dx \leq \int_{\Omega} \delta(t)^{m'/m} (Q(t, x, u_t(t, x), u_t(t, x)))dx = \delta(t)^{m'/m} \mathcal{D}u(t) < \infty$$

a.e. in I , since $\delta \in L^1_{\text{loc}}(I)$ and $\mathcal{D}u \in L^1(I)$ for Lemma (ii).

5. The Bow-up case

Consider the abstract evolution equation

$$(5.1) \quad [P(u'(t))]' + A(u(t)) + Q(t, u'(t)) - F(u(t)) = 0,$$

with $t \in I = [T, \infty)$, and $\langle F(u), u \rangle_X \geq 0$ as before, but in equation (5.1) we now have the minus sign for $F(u)$, a *different* problem than (4.1).

In any case we take, as before for (4.1),

$$P : V \rightarrow V', \quad A : W \rightarrow W', \quad F : X \rightarrow X',$$

and we assume here, moreover, that $X \subset V$ with continuous inclusion. For instance, in the case of the wave system (4.12), this reduces to the request that $p \geq 2$, so that now $2 \leq m \leq p$.

As before, assume (S1)–(S3), except that in (S2) we require only that $\mathcal{A}(u) \geq 0$ in U . Of course now

$$\mathcal{E}u(t) = \mathcal{P}^*(u'(t)) + \mathcal{A}(u(t)) - \mathcal{F}(u(t)),$$

and again we take (a), (b) and (c) of Section 4 as the definition of a strong solution of (5.1). Explicitly by (c)

$$t \mapsto \mathcal{E}u(t) + \int_T^t \mathcal{D}u(s)ds \quad \text{is non-increasing in } I.$$

Then under a (weaker) version than (S4) – since no longer lower bounds on Q are needed – together with (S5) and a *new* assumption replacing (S6), we shall show that, subject to an appropriate condition for $\delta = \delta(t)$, which is related to the *k conditions* of the main stability Theorem 4.1, a *strong solution* of (5.1) must fail at some finite value of time t_0 with $T < t_0 < \infty$.

The new conditions (S4) and (S6) are the following:

(S4)' There are an exponent $m > 1$ and a function δ of class $C^1(I \rightarrow \mathbb{R}_0^+)$ such that along any strong solution $u = u(t)$ of (5.1)

$$\|Q(t, u'(t))\|_{X'} \leq \delta(t)^{1/m} \mathcal{D}u(t)^{1/m'} \quad \text{a.e. in } I.$$

(S6)' There are positive constants c, q and an exponent $\nu > 1$ such that

$$\begin{aligned} \langle A(u), u \rangle_W &\leq q\mathcal{A}(u) \quad \text{in } W \\ c\|P(v)\|_{V'}^{\nu'} &\leq (q+1)\langle P(v), v \rangle_V - q\mathcal{P}(v) \quad \text{in } V; \end{aligned}$$

moreover, for every $\tau > 0$ there are $c_1 = c_1(\tau) > 0, c_2 = c_2(\tau) > 0$ such that

$$c_1\mathcal{F}(u) \leq c_2\|u\|_X^p \leq \langle F(u), u \rangle_X - q\mathcal{F}(u) \quad \text{whenever } u \in U \text{ and } \mathcal{F}(u) \geq \tau.$$

Note that for $A = -\Delta$ then (S6)'₁ holds with $q = 2$. Thus in the wave system case, where $P = I$ and $V = [L^2(\Omega)]^N$, we have $\mathcal{P}(v) = \frac{1}{2}\|v\|_2^2$, so that $\nu = 2$, and as noted above $q = 2$. Hence (S6)'₂ reduces to

$$c\|v\|_2^2 \leq 3\|v\|_2^2 - 2\frac{1}{2}\|v\|_2^2 = 2\|v\|_2^2,$$

or (S6)'₂ holds with $c = 2$. Moreover, in the concrete system with $f(x, u) = |u|^{p-2}u$, $p > 1$, we have $F(u) = f(\cdot, u(\cdot))$ and $\mathcal{F}(u) = \frac{1}{p}\|u\|_p^p$, since $X = [L^p(\Omega)]^N$. For the validity of (S6)'₃ we need only that c_1, c_2 are constants and

$$0 < c_1 \leq pc_2 \leq p - q, \quad p > q.$$

In the wave system case in which $q = 2$ necessarily $p > 2$ and

$$0 < c_1 \leq pc_2 \leq p - 2.$$

Note also that in the wave system case the stabilizing term is $-\Delta$ and the destabilizing term is $-f(x, u)$. Thus, in the blow-up problem, the destabilizing term must be stronger than the stabilizing term, i.e. $p > 2$.

Theorem 5.1. Assume (S1)–(S3), (S4)', (S5) and (S6)'. Suppose also that

$$(5.2) \quad 1 < \nu \leq m < p,$$

$$(5.3) \quad \int^{\infty} \frac{\varrho(t)}{\max\{1, \varrho(t)\}^{1+\vartheta}} dt = \infty,$$

for all $\vartheta > 0$ sufficiently small, where $\varrho = \delta^{-1/(m-1)}$, and

$$(5.4) \quad \delta'(t) = o(\delta(t)) \quad \text{as } t \rightarrow \infty.$$

Then there can be no strong solution in I of (5.1) with $\mathcal{E}u(T) < 0$.

Remarks. 1. In the wave system case, condition (5.2) reduces to

$$2 \leq m < p,$$

which fits perfectly well with (S6)'.

2. As before, $\mathcal{E}u$ is non-increasing on I so that

$$\mathcal{E}u(t) \leq \mathcal{E}u(T) < 0$$

by the hypothesis of Theorem 5.1.

3. As an example, take $T = 1$, $\gamma \in \mathbb{R}$ and $\delta(t) = t^\gamma$. Thus $\delta'(t) = o(\delta(t))$ as $t \rightarrow \infty$.

Also

$$\varrho(t) = t^{-\gamma/(m+1)},$$

so condition (5.3) forces

$$-\infty < \gamma \leq m - 1$$

(choose $\vartheta > 0$ so small that $\vartheta|\gamma| < m - 1$ when $\gamma < 0$). This request is reasonable since the damping term must not be too large if one is to have blow-up.

4. The assertion of Theorem 5.1 says that there is a time t_0 with $T < t_0 < \infty$ such that

$$\|u(t)\|_U + \|u'(t)\|_V \rightarrow \infty \quad \text{as } t \rightarrow t_0.$$

The proof of Theorem 5.1 is too long to be given in this short presentation.

Outline of the proof of Theorem 5.1. Assume for contradiction that there is a strong solution u of (5.1) in the entire I . From the form of the energy we find

$$\mathcal{F}u(t) \geq -\mathcal{E}u(t) \geq -\mathcal{E}u(T) = \tau > 0,$$

so that (S6)'₃ applies.

Now take $\phi = u \in K$ in the distribution identity (b). Thus along u

$$\begin{aligned} \frac{d}{dt} \langle P(u'(t), u(t)) \rangle_V &= \langle P(u'(t), u'(t)) \rangle_V + \langle F(u(t)) - A(u(t)) - Q(t, u'(t)), u(t) \rangle_U \\ &\quad + q \{ \mathcal{P}^*(u'(t)) + \mathcal{A}(u(t)) - \mathcal{F}(u(t)) - \mathcal{E}u(t) \} \\ &= (q+1) \{ \langle P(u'(t), u'(t)) \rangle_V - q \mathcal{P}(u'(t)) \} + q \mathcal{A}(u(t)) - \langle A(u(t), u(t)) \rangle_U \\ &\quad + \langle F(u(t), u(t)) \rangle_U - q \mathcal{F}(u(t)) - q \mathcal{E}u(t) - \langle Q(t, u'(t)), u(t) \rangle_U \\ &\geq C \|P(u'(t))\|_{V'}^{\nu'} + C_2 \|u(t)\|_X^p - q \mathcal{E}u(t) - \langle Q(t, u'(t)), u(t) \rangle_U, \end{aligned}$$

thanks to the main structural assumptions. The bad term in the last expression is clearly

$$-\langle Q(t, u'(t)), u(t) \rangle_U.$$

Treating this by (S4)' gives

$$\langle Q(t, u'(t)), u(t) \rangle_U \leq \|Q(t, u'(t))\|_{X'} \cdot \|u(t)\|_X \leq \delta(t)^{1/m} \mathcal{D}u(t)^{1/m'} \|u(t)\|_X.$$

After some further calculation, we finally get the following conclusion. Put

$$H(t) = -\mathcal{E}u(T) + \int_T^t \mathcal{D}u(s) ds \quad \text{non-decreasing in } I$$

and

$$Z(t) = \lambda H(t)^{1-\alpha} + \varrho(t) \langle P(u'(t), u(t)) \rangle_V.$$

Then for $\alpha > 0$ sufficiently small and $\lambda > 0$ sufficiently large, we find that $Z(t) > 0$ in I and a.e. in I

$$Z'(t) \geq \text{Const.} \frac{\varrho(t)}{\max\{1, \varrho(t)\}^{1+\vartheta}} [Z(t)]^{1+\vartheta},$$

with $\vartheta = \alpha/(1-\alpha)$. In turn

$$\frac{Z'(t)}{[Z(t)]^{1+\vartheta}} \geq R(t) := \text{Const.} \frac{\varrho(t)}{\max\{1, \varrho(t)\}^{1+\vartheta}}.$$

Rewrite this as

$$\frac{d}{dt} \left(-\frac{1}{\vartheta Z^\vartheta} \right) \geq R(t),$$

and integrate from T to t to obtain

$$-\frac{1}{\vartheta Z(t)^\vartheta} + \frac{1}{\vartheta Z(T)^\vartheta} \geq \int_T^t R(s) ds,$$

that is, since $Z(t) > 0$,

$$\int_T^t R(s)ds \leq \frac{1}{\vartheta Z(T)^\vartheta},$$

which contradicts condition (5.3). Hence the solution u cannot be continued for all $t \in I$, and this completes the sketch of the proof of Theorem 5.1.

In particular, if t_1 is defined by

$$\int_T^{t_1} R(s)ds = \frac{1}{\vartheta Z(T)^\vartheta},$$

then, since Z is increasing on I , we see that

$$\lim_{t \rightarrow t_2} Z(t) = \infty \quad \text{for some } t_2 \in (T, t_1].$$

Consequently, if the solution u can be continued for $t \in [T, t_2)$, then necessarily

$$\lim_{t \rightarrow t_2} \|u(t)\|_X = \infty.$$

In turn, since $U \subset X$, also

$$\lim_{t \rightarrow t_2} \|u(t)\|_U = \infty.$$

Hence, at least formally, there must exist $t_0 \leq t_2 \leq t_1$ such that

$$\lim_{t \rightarrow t_0} \{\|u(t)\|_U + \|u'(t)\|_V\} = \infty.$$

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