Asymptotic Stability for Nonlinear Parabolic Systems

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1. Introduction

The problem of asymptotic stability for second order ordinary differential equations is well–known in the literature. Recently, various extensions of this work have been given for the case of second order hyperbolic systems (see [1–5]). On the other hand, the situation for parabolic systems has received much less discussion, so that a study of this problem seems worthwhile.

We consider specifically systems of the form

$$(1.1) \qquad \begin{cases} A(t)|u_t|^{m-2}u_t = \Delta u - f(x,u), & (t,x) \in J \times \Omega, \\ u(t,x) = 0, & (t,x) \in J \times \partial\Omega, \end{cases}$$

where $J = [0,\infty)$ and Ω is a bounded open subset of $\mathbb{R}^n$. The values of u are taken in $\mathbb{R}^N$, $N \geq 1$, and

$$A \in C(J \to \mathbb{R}^{N\times N}), \qquad f \in C(\Omega \times \mathbb{R}^N \to \mathbb{R}^N),$$

with $m > 1$ a fixed exponent. In order that (1.1) be parabolic, it is necessary to have

$$(1.2) \qquad \big(A(t)v, v\big) > 0 \qquad \text{for all } t \in J \text{ and } v \in \mathbb{R}^N \setminus \{0\},$$

where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^N$; in particular note that A need not be symmetric. Moreover, we assume that f represents a *restoring force* derivable from a potential F, namely,

$$(1.3) \qquad \big(f(x,u), u\big) \geq 0$$

4.9and

$$(1.4) \qquad f(x,u) = \frac{\partial F}{\partial u}(x,u),$$

where $F \in C^1(\Omega \times \mathbb{R}^N \to \mathbb{R})$ and $F(x, 0) = 0$. The last condition is a normalization which can be assumed without loss of generality.

Note specifically that the relation (1.4) is redundant when $N = 1$, since in this case F can be obtained simply by integrating the force with respect to u.

If we take $m = 2$ then (1.1) reduces to the strongly coupled parabolic system

$$A(t)u_t = \Delta u - f(x, u),$$

or simply to

$$u_t = \Delta u - f(x, u),$$

when $A(t) = I$.

A canonical example of the type of functions f which we contemplate here is

$$f(x, u) = V(x)|u|^{p-2}u, \tag{1.5}$$

where $1 < p < \infty$, $m \le \max\{p,\ 2n/(n-2)\}$ and $V \in C(\overline{\Omega} \to \mathbb{R}_0^+)$.

In the context of problem (1.1) the question of asymptotic stability of the rest state is best considered in terms of the natural energy associated with solutions of the system, namely,

$$Eu(t) = \int_\Omega \{\tfrac{1}{2}|Du(t,x)|^2 + F(x, u(t,x))\}dx.$$

In particular, the rest field $u(t, x) \equiv 0$ will be called *asymptotically stable (in the mean)*, if and only if

$$\lim_{t\to\infty} Eu(t) = 0 \qquad \text{for all solutions } u = u(t,x) \text{ of (1.1)}.$$

In this formulation we have tacitly assumed that solutions are classical, but for a useful theory one must actually consider solutions in a wider class of functions. We treat this question in Section 2. Our main result is formulated in Section 3, and several applications are given in Section 4, including for example the mean curvature equation

$$A(t)u_t = \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) \tag{1.6}$$

and the degenerate Laplace system

$$A(t)u_t = \operatorname{div}(|Du|^{s-2}Du), \qquad s > 1. \tag{1.7}$$

We assume throughout that the reader is familiar with the results of reference [5], particularly Sections 2, 3 and 6.

The linear case

An important special case of (1.1) and (1.5) occurs when $p = m = 2$ and $A(t) = a(t)t^\alpha$, with $a \in C(J \to \mathbb{R}_0^+)$ and $\alpha \in \mathbb{R}$. The system then becomes (with $N = 1$)

$$a(t)t^\alpha u_t = \Delta u - V(x)u.$$

If a is bounded on J, and V is as above, then we prove that the rest state is asymptotically stable if $\alpha \le 1$. On the other hand, if a is also *bounded from zero* in J, then for $\alpha > 1$ there exist solutions which approach non–zero functions $\psi = \psi(x)$ as $t \to \infty$.

2. Definition of Solutions

To provide an appropriate definition for solutions of (1.1) it is convenient first to introduce the elementary bracket pairing in $\Omega \subset \mathbb{R}^n$

$$\langle \varphi, \psi \rangle = \langle \varphi, \psi \rangle(t) = \int_\Omega (\varphi, \psi) dx, \qquad \text{where } \varphi,\, \psi :\ J \times \Omega \to \mathbb{R}^N,$$

this being a well–defined real function of time for all $t \in J$ such that $(\varphi, \psi) \in L^1(\Omega)$. We write for simplicity

$$L^p = \left[L^p(\Omega)\right]^N, \qquad X = \left[W_0^{1,2}(\Omega)\right]^N,$$

where $p > 1$; these spaces are endowed respectively with the natural norms

$$\|\varphi\|_{L^p}, \qquad \|\varphi\|_X = \|D\varphi\|_{L^2} = \|D\varphi\|.$$

Now define $K' = C(J \to X)$ and

$$K = \{\phi \in K' \ :\ E\phi \text{ is locally bounded on } J\},$$

where $E\phi$ is the *total energy of the field* ϕ, that is

$$E\phi = E\phi(t) = \tfrac{1}{2}\|D\phi\|^2 + \int_\Omega F(x, \phi(t,x)) dx,$$

it being tacitly assumed that $F(\cdot, \phi(t, \cdot)) \in L^1_{\text{loc}}(\Omega)$ for all $t \in J$.

We can now give our principal definition. A *strong solution of* (1.1) is a function $u \in K$ which is weakly differentiable with respect to t in $J \times \Omega$ and which satisfies the following *two conditions*:

(A) *Conservation Law*

(i) $$\mathcal{R}(t) \in L^1_{\rm loc}(J),$$

(ii) $$Eu\Big]_0^t = -\int_0^t \mathcal{R}(s)ds \quad \text{for all } t \in J,$$

where $\mathcal{R}(t) = \langle A(t)u_t, |u_t|^{m-2}u_t\rangle$.

(B) *Distribution Identity*

$$\int_0^t \{\langle Du, D\phi\rangle + \langle A(s)u_t, |u_t|^{m-2}\phi\rangle + \langle f(\cdot,u),\phi\rangle\}ds = 0$$

for all $t \in J$ and $\phi \in K$.

For various comments on this definition in the related context of damped wave systems, see [5].

An important issue is to determine a class of functions $A(t)$ and $f(x,u)$ for which the second and third terms in the distribution identity are well–defined, i.e. satisfy

$$\langle A(t)u_t, |u_t|^{m-2}\phi\rangle,\ \langle f(\cdot,u),\phi\rangle \in L^1_{\rm loc}(J). \tag{2.1}$$

Letting $r = 2n/(n-2)$ be the Sobolev exponent for the space X ($r = \infty$ if $n = 1$; $2 < r < \infty$ if $n = 2$, since Ω is bounded), we make the natural hypothesis

$$|f(x,u)| \le \text{Const.}\,[1 + |u|^{p-1}], \qquad p > 1. \tag{2.2}$$

Moreover, if $n \ge 3$ and $p > r$, we suppose there are constants $\kappa > 0$ and $\kappa_0 \ge 0$ such that

$$\big(f(x,u),u\big) \ge \kappa|u|^p - \kappa_0|u|. \tag{2.3}$$

Also, for $t \in J$, let

$$H(t) = |A(t)|, \qquad h(t) = \min_{|v|=1}\big(A(t)v,v\big); \tag{2.4}$$

by (1.2) it is clear that both $h(t)$ and $H(t)$ are positive and continuous on J.

Under assumptions (2.2)–(2.3), the conditions (2.1) then hold provided that

$$m \le \max\{p,\ r\}, \qquad \delta \in L^1_{\rm loc}(J), \tag{2.5}$$

where $\delta = \delta(t) = H^m/h^{m-1}$. This is proved almost exactly as in Section 2 of [5]. In particular, Lemmas 2.1 and 2.2 of [5] hold exactly as stated, so that $\langle f(\cdot,u),\phi\rangle$ is locally bounded on J when u, $\phi \in K$. Hence this term in (B) is well–defined.

To show that the first part of (2.1) also holds, we observe by Schwarz' inequality and (2.4) that, for $(t,x)\in J\times\Omega$,

$$
\begin{aligned}
|\big(A(t)u_t, |u_t|^{m-2}\phi\big)| &\le H(t)|u_t|^{m-1}|\phi| \le H(t)|u_t|^{2/m'}|u_t|^{(m-2)/m'}|\phi| \\
&\le H(t)\,h(t)^{-1/m'}\big(A(t)u_t,u_t\big)^{1/m'}|u_t|^{(m-2)/m'}|\phi| \\
&= \delta(t)^{1/m}\big(A(t)u_t,|u_t|^{m-2}u_t\big)^{1/m'}|\phi|,
\end{aligned}
\tag{2.6}
$$

where m' is the Hőlder conjugate of m. In turn, from Hőlder's inequality and the definition of $\mathcal{R}(t)$,

$$
|\langle A(t)u_t, |u_t|^{m-2}\phi\rangle| \le \delta(t)^{1/m}\cdot\mathcal{R}(t)^{1/m'}\cdot\|\phi\|_{L^m}, \tag{2.7}
$$

for $t\in J$. Hence by Hőlder's inequality again, together with the facts that $\delta(t)$ and $\mathcal{R}(t)$ are in $L^1_{\mathrm{loc}}(J)$, and that $\|\phi\|_{L^m}$ is in $L^\infty_{\mathrm{loc}}(J)$, see Lemma 2.1 of [5], the condition $(2.1)_1$ follows at once.

Remark. When $N=1$, or when $A(t)$ is a multiple of the identity matrix, we have $h(t)=H(t)=|A(t)|$, so that $\delta(t)=|A(t)|$. Moreover, if either $n=1$ or 2, the restriction on m reduces simply to $1<m<\infty$.

3. Asymptotic Stability

We can now give the main result of the paper.

THEOREM 3.1. *Let* (2.2), (2.3) *and* (2.4) *hold. Suppose there exists a non–negative continuous function k on J, such that*

$$
k\notin L^1(J), \tag{3.1}
$$

$$
\liminf_{t\to\infty}\int_0^t \delta\, k^m ds \Big/ \left(\int_0^t k ds\right)^m < \infty. \tag{3.2}
$$

Then the rest state $u\equiv 0$ of (1.1) *is asymptotically stable.*

Proof. The proof is essentially the same as for Theorem 3.1 of [5], with the main difference that (3.18) in that paper is here replaced by the relation

$$
\int_T^t \big(\|Du\|^2 + \langle f(\cdot,u),u\rangle + \langle A(s)u_t, |u_t|^{m-2}u\rangle\big)k(s)ds = 0. \tag{3.3}
$$

Because of the simpler form of this identity, it is no longer necessary to assume that k is of bounded variation, or to make use of the analogue of Lemma 3.3 of [5]. On the other hand, the analogues of Lemmas 3.1 and 3.4 continue to hold, as is easily seen. Moreover, for the analogue of Lemma 3.2 we have specifically, see (2.7),

$$\int_T^t |\langle A(s)u_t, |u_t|^{m-2}u\rangle| k(s)ds \le \left(\int_T^t \delta\, k^m ds\right)^{1/m} \left(\int_T^t \mathcal{R}(s)ds\right)^{1/m'} \|u\|_{L^m}$$
$$\le \varepsilon(T)\left(\int_0^t \delta\, k^m ds\right)^{1/m}$$

for $t \ge T \ge 0$, and $\varepsilon(T) \to 0$ as $T \to \infty$, as in [5]. This being the case, we get from (3.3), exactly as in the proof of Theorem 3.1 of [5],

$$-\alpha\int_T^t kds + \varepsilon(T)\left(\int_0^t \delta\, k^m ds\right)^{1/m} \ge 0$$

for $t \ge T \ge 0$. By (3.2) there is a sequence $t_i \nearrow \infty$ and a number $M > 0$ such that

$$\int_0^{t_i} \delta\, k^m ds \le \left(M\int_0^{t_i} kds\right)^m.$$

Consequently, taking T so large that $\varepsilon(T)M \le \alpha/2$, we obtain

$$\frac{\alpha}{2}\int_0^T kds - \frac{\alpha}{2}\int_T^{t_i} kds \ge 0,$$

which yields an immediate contradiction with (3.1) when $t_i \nearrow \infty$. This completes the proof.

4. Applications

Important special cases of Theorem 3.1 occur when $k = 1$ and $k = \delta^{1/(1-m)}$. In the first case, the assumptions (3.1) and (3.2) reduce to the single condition

$$\liminf_{t\to\infty} \frac{1}{t^m}\int_0^t \delta(s)ds < \infty, \tag{4.1}$$

and in the second to

$$\int_0^\infty \delta^{1/(1-m)}dt = \infty. \tag{4.2}$$

An interesting extension of the results occurs if the term $A(t)|u_t|^{m-2}u_t$ in (1.1) is replaced by a general function of the form $Q(t,x,u,u_t)$. In this case it is necessary to define $\mathcal{R}(t) = \langle Q(t,\cdot,u,u_t),u_t\rangle$ in condition (A) of Section 2 and to replace the principal conditions (2.4), (2.5) by – see (2.6) –

$$|Q(t,x,u,v)| \le \hat{\delta}(t)^{1/m} \cdot \big(Q(t,x,u,v),v\big)^{1/m'}, \tag{4.3}$$

where $1 < m \le \max\{p,\ r\}$ and $\hat{\delta} \in L^1_{\mathrm{loc}}(J)$. Then Theorem 3.1 continues to hold, provided in (3.2) we change $\delta\, k^m$ to $\hat{\delta}\, k^m$.

Finally, the Laplace operator in (1.1) can be replaced by various elliptic operators without affecting the results, as shown in Section 6 of [5]. Cases of particular importance arise for the mean curvature equation (1.6) and the degenerate Laplace system (1.7).

In the case of (1.6), for instance, we take $X = W_0^{1,1}(\Omega)$. Condition (2.2) holds trivially since $f \equiv 0$, while on the other hand (2.3) fails for all $p > 1$. It is therefore necessary to replace the first condition of (2.5) by $m \le r$, where r is the Sobolev exponent for $W_0^{1,1}(\Omega)$, namely $r = n/(n-1)$. Obviously $m = 2$ in the present case, which gives the condition $2 \le n/(n-1)$. This means that our stability results apply to (1.6) exactly when $n = 1$ or $n = 2$. More precisely, we have the following conclusion.

Theorem 4.1. *Let u be a strong solution of equation* (1.6)*, and suppose either $n = 1$ or $n = 2$. Assume there exists a non–negative continuous function k on J such that* (3.1) *and* (3.2) *hold with $m = 2$ and $\delta(t) = A(t)$. Then*

$$\lim_{t\to\infty} \int_\Omega \sqrt{1+|Du|^2}\, dx = |\Omega|. \tag{4.4}$$

Proof: Equation (1.6) corresponds to the case

$$A(w) = w/\sqrt{1+|w|^2}, \qquad G(w) = \sqrt{1+|w|^2} - 1, \qquad s = 1$$

in [5], Section 6.3. Since $f \equiv 0$, the total energy of the field u is easily seen to be given by

$$Eu(t) = \int_\Omega G(Du(t,x))\, dx.$$

As in Section 3 we then obtain $Eu(t) \to 0$ as $t \to \infty$, which is equivalent to the stated conclusion. This result also implies that $\|u\|_{L^2} \to 0$ as $t \to \infty$.

A similar argument also applies to the system (1.7), giving the corresponding

Theorem 4.2. *Let u be a strong solution of the system* (1.7), *where*

$$s \geq 2n/(n+2).$$

Assume there exists a non–negative continuous function k on J such that (3.1) *and* (3.2) *hold with $m = 2$ and $\delta(t) = H^2/h$. Then*

$$\lim_{t\to\infty} \|Du\|_{L^s} = 0. \tag{4.5}$$

When $A(t)$ is uniformly bounded on J, the conditions (3.1) and (3.2) are satisfied with $k = 1$. In particular, then, the rest state for (1.6) and for (1.7) is asymptotically stable in the classical case $A(t) \equiv I$, provided that $n = 1$ or $n = 2$ for (1.6), or $s \geq 2n/(n+2)$ for (1.7).

Another case of interest occurs for the modified equations

$$\frac{u_t}{\sqrt{1+|Du|^2}} = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \tag{1.6$'$}$$

and

$$\frac{u_t}{\sqrt{1+|Du|^2}} = \operatorname{div}(|Du|^{s-2}Du) \tag{1.7$'$}$$

(with $N = 1$), whose left hand sides represent the normal velocity of the surface $u = u(t,x)$ as a function of time in the $(n+1)$–dimensional (x,u) space. These equations cannot be put in the form (1.6), (1.7) because the coefficient $1/\sqrt{1+|Du|^2}$ depends on both t and x. On the other hand, by taking

$$Q(t,x,u,v) = \frac{v}{\sqrt{1+|Du(t,x)|^2}},$$

as at the beginning of this section, we find that

$$|Q(t,x,u,v)| \leq \big(Q(t,x,u,v)v\big)^{1/2}.$$

Thus (4.3) holds with $m = 2$ and $\hat{\delta} \equiv 1$. In turn, the conclusion (4.4) of Theorem 4.1 holds for $(1.6)'$ when $n = 1$ or $n = 2$, and the conclusion (4.5) of Theorem 4.2 holds for $(1.7)'$ when $s \geq 2n/(n+2)$.

The linear case

Consider the problem

$$(4.6)\qquad \begin{cases} a(t)t^\alpha u_t = \Delta u - V(x)u & \text{in } J\times\Omega, \\ u(t,x)=0 & \text{on } J\times\partial\Omega, \end{cases}$$

where $N = 1$ for simplicity, Ω is a bounded open subset of $\mathbb{R}^n$, and $a \in C(J \to \mathbb{R}_0^+)$, $V \in C(\overline{\Omega} \to \mathbb{R}^+)$. Since $N = 1$ we have $\delta(t) = H(t) = a(t)t^\alpha$ in (2.5); it is convenient here to take $J = [1,\infty)$ in order to avoid the singularity at $t = 0$ when $\alpha < 0$. Then, with $m = p = 2$, it is clear that (2.2) and $(2.5)_1$ are satisfied. Moreover, assuming that

$$(4.7)\qquad a(t)\le C \qquad \text{in } J,$$

we get $(2.5)_2$.

It follows now, either from (4.1) or (4.2), that *the rest state is asymptotically stable for* (4.6) *whenever* (4.7) *holds and* $\alpha \le 1$. [Actually, using (4.2), we find that asymptotic stability holds for $\alpha \le 1$ even when (4.7) is replaced by $a(t) \le C\log t$.]

When $\alpha > 1$, neither (4.1) nor (4.2) applies. In fact, in this situation solutions of (4.6) *do not* in general approach zero as $t \to \infty$. To illustrate this case, let φ_k be the k^{th} eigenfunction of $-\Delta + V(x)$ in Ω, with Dirichlet boundary conditions.

We say that a function

$$\psi = \psi(x) \in Y \equiv \operatorname{span}\{\varphi_k\}_{k=1}^\infty$$

is *attainable* if there exists a solution $u \in K$ of (4.6) such that

$$(4.8)\qquad \lim_{t\to\infty} \|u(t) - \psi\|_{L^2} = 0.$$

Theorem 4.3. *Suppose $\alpha > 1$ and also that $a(t) \ge 1/C$ for all $t \in J$. Then every function $\psi \in Y$ is attainable for problem* (4.6)*. In turn, the set of attainable functions is dense in L^2.*

Proof: We first show that every eigenfunction φ_k is attainable. For this purpose consider the function

$$u_k(t,x) = w_k(t)\varphi_k(x),$$

which satisfies (4.6) if and only if w_k is a solution of the ordinary differential equation

$$(4.9)\qquad a(t)t^\alpha w' + \mu_k w = 0, \qquad t\in J,$$

where $\mu_k > 0$ is the eigenvalue associated to φ_k. By integration we get

$$w(t) = \text{ Const. }\exp\left(-\int_1^t \frac{\mu_k}{a(s)s^\alpha}ds\right).$$

Since $\alpha > 1$ and $a(s) \geq 1/C$ in J, the integral is convergent, whence

$$\lim_{t\to\infty} w(t) \quad \text{exists and is finite.}$$

It follows that the set of attainable limits of solutions of (4.9) is all of $\mathbb{R}$. Hence for the particular solution w_k of (4.9) which has limit value one at infinity, we get

$$\lim_{t\to\infty} \|u_k(t,\cdot) - \varphi_k(\cdot)\|_{L^2} = 0.$$

Finally, using the linearity of (4.6), we obtain (4.8) for every $\psi \in Y$.

Acknowledgment. P. Pucci is a member of *Gruppo Nazionale di Analisi Funzionale e sue Applicazioni* of the *Consiglio Nazionale delle Ricerche.* This research has been partly supported by the Italian *Ministero della Università e della Ricerca Scientifica e Tecnologica.*

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