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Existence, Stability and Blow–up for Dissipative Evolution Equations

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1 INTRODUCTION

The problem of stability and blow–up for dissipative evolution equations will be treated by Lyapunov–type methods. The discussion will be carried out particularly in the context of evolution operators in a Banach space, with special care given to an appropriate definition of solution, and with the specific examples of degenerate damped wave equations and degenerate parabolic equations as principal applications.

Our treatment is expository in intent, based principally on the references [3, 7, 9, 10, 17, 18, 20]. Here we discuss mainly simplified versions of the results, in order to show the main ideas of the theory and to avoid technicalities.

2 GENERAL SETTING

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The abstract evolution equation which we consider are of the type

$$[P(u_t)]_t + Q(t, u_t(t)) + A(u(t)) = F(u(t)), \quad t \in J = [0, \infty). \tag{2.1}$$

Simple examples, when $u : J \times \Omega \to \mathbb{R}$, with Ω an open bounded subset of $\mathbb{R}^n$, are the following:

1. The wave equation: when $P = I$, $Q = 0$, $A = -\Delta$,

$$u_{tt} - \Delta u = f(x, u);$$

2. The wave equation with linear dissipation: as above, with $Q(t, v) = bv$ and $b > 0$,

$$u_{tt} + bu_t - \Delta u = f(x, u);$$

3. The parabolic equation: when $P = 0$, $Q(v) = v$, $A = -\Delta$,

$$u_t = \Delta u + f(x, u).$$

Of course the operators corresponding to these concrete cases must be understood in the sense of Nemitsky, as being defined by appropriate distribution relations. We shall discuss this in detail a little later.

First let us clarify the meaning of (2.1) as an abstract evolution equation. In particular we suppose that

$$P : V \to V', \quad A : W \to W', \quad F : X \to X'$$

for real Banach spaces V, W, X, and their dual spaces V', W', X'. Moreover, we suppose that P, A, F are the Fréchet derivatives of real C^1 potentials

$$\mathcal{P} : V \to \mathbb{R}, \quad \mathcal{A} : W \to \mathbb{R}, \quad \mathcal{F} : X \to \mathbb{R},$$

with $\mathcal{P}(0) = \mathcal{A}(0) = \mathcal{F}(0) = 0$ (normalization); thus from the definition of derivative we immediately get, in particular, that

$$\mathcal{F}(u) = \int_0^1 \langle F(\tau u), u \rangle_X d\tau,$$

where $\langle x', x \rangle_X = x'(x)$ for all $x \in X$, $x' \in X'$; and so on. Finally

$$Q : J \times Y \to X',$$

with $Y \hookrightarrow V$ continuously as Banach spaces. Some mild simplifications from the original papers have been made in order to clarify this exposition.

EXAMPLE 1. $A = -\Delta$, $W = H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^n$. By definition, for $u \in H_0^1(\Omega)$, the expression $-\Delta u$ denotes the element w' of $[H_0^1(\Omega)]'$ such that

$$\langle w', \varphi \rangle_W \equiv w'(\varphi) = \int_\Omega (Du, D\varphi) dx \quad \text{for all } \varphi \in H_0^1(\Omega).$$

It is easy to check that A is the Fréchet derivative of the potential

$$\mathcal{A}(u) = \tfrac{1}{2}\int_\Omega |Du|^2 dx = \tfrac{1}{2}\|u\|_W^2.$$

Similarly, for the degenerate s–Laplace operator $A = -\Delta_s = -\mathrm{div}(|Du|^{s-2}Du)$, we take $W = W_0^{1,s}(\Omega)$ and $-\Delta_s u$ as the element w' of W' such that

$$\langle w', \varphi\rangle_W = \int_\Omega |Du|^{s-2}(Du, D\varphi)dx \quad \text{for all } \varphi \in W_0^{1,s}(\Omega).$$

It follows at once that

$$\mathcal{A}(u) = \frac{1}{s}\int_\Omega |Du|^s dx = \frac{1}{s}\|u\|_W^s.$$

EXAMPLE 2. Let P be a symmetric operator from a real Hilbert space V into V', that is

$$\langle Pv, \varphi\rangle_V = \langle P\varphi, v\rangle_V \quad \text{for all } v,\ \varphi \in V.$$

It is easy to prove that P must then be linear and also, by the uniform boundedness principle, continuous. Moreover, as is readily verified, by virtue of the symmetry of P we have the important formula

$$\mathcal{P}(v) = \tfrac{1}{2}\langle Pv, v\rangle_V.$$

As a special case, let $V = [L^2(\Omega)]^N$ and define the Riesz identity $I : V \to V'$ by

$$\langle Iv, \varphi\rangle_V = \int_\Omega (v, \varphi)dx.$$

Then $\mathcal{I}(v) = \tfrac{1}{2}\|v\|^2_{[L^2(\Omega)]^N}$. By abuse of notation one frequently writes $\langle v, \varphi\rangle_V$ instead of $\langle Iv, \varphi\rangle_V$.

EXAMPLE 3. Let $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ be continuous. Define $\Phi(x, u) = \displaystyle\int_0^u f(x, z)dz$, so that $f(x, u) = \dfrac{\partial\Phi}{\partial u}(x, u)$ for all $x \in \overline{\Omega}$, $u \in \mathbb{R}$. For vector functions $f : \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N$ this definition is an assumption. Assume that

$$|f(x, u)| \le \text{ Const.}\,(1 + |u|^{p-1}), \quad p > 1.$$

Then we can take $X = L^p(\Omega)$, and for $u \in X$ define $F(u)$ to be the element x' (Nemitsky operator) of $X' = L^{p'}(\Omega)$ such that

$$\langle x', \varphi\rangle_X \equiv w'(\varphi) = \int_\Omega f(x, u(x))\varphi(x)dx \quad \text{for all } \varphi \in L^p(\Omega).$$

(Here p' is the Hőlder conjugate of p.) That this is well–defined follows from the calculation

$$|f(x, u(x))|^{p'} = |f(x, u(x))|^{p/(p-1)} \le \text{ Const.}\,(1 + |u(x)|^p), \quad \text{in } \Omega,$$

so that $f(\cdot, u(\cdot)) \in L^{p'}(\Omega)$ for each $u \in L^p(\Omega)$. Furthermore we calculate easily that

$$\begin{aligned}\mathcal{F}(u) &= \int_0^1 \int_\Omega f(x, \tau u(x)) u(x) dx d\tau = \int_0^1 \int_\Omega \frac{d\Phi}{d\tau}(x, \tau u(x)) dx d\tau \\ &= \int_\Omega \int_0^1 \frac{d\Phi}{d\tau}(x, \tau u(x)) d\tau dx = \int_\Omega \Phi(x, u(x)) dx.\end{aligned}$$

Note that $\mathcal{F}(u) = \|u\|^p_{L^p(\Omega)}/p$ when $f(u) = |u|^{p-2}u$. Moreover of course $F \in C(X \to X')$.

EXAMPLE 4. For the continuous function $\tilde{Q} : J \times \mathbb{R} \to \mathbb{R}$, defined by

$$\tilde{Q}(t, v) = b(t)|v|^{m-2}v, \quad m > 1,$$

the corresponding (Nemitsky) operator $Q : J \times Y \to Y'$ is given by

$$\langle Q(t, v), \varphi \rangle_Y = \int_\Omega \tilde{Q}(t, v(x)) \varphi(x) dx$$

for $Y = L^m(\Omega)$, $Y' = L^{m'}(\Omega)$. Various more general functions $\tilde{Q}$ can also be allowed; see [9, 17].

Let us now return to the abstract problem (2.1), and give meaning to the idea of a solution. To begin with, we note that, formally,

$$\begin{aligned}\int_0^t \langle [P(u_t(\tau))]_t, \varphi(\tau) \rangle_V d\tau &= \int_0^t \left\{ \frac{d}{d\tau} \langle P(u_t(\tau)), \varphi(\tau) \rangle_V - \langle P(u_t(\tau)), \varphi_t(\tau) \rangle_V \right\} d\tau \\ &= \langle P(u_t(\tau)), \varphi(\tau) \rangle_V \Big|_0^t - \int_0^t \langle P(u_t(\tau)), \varphi_t(\tau) \rangle_V d\tau,\end{aligned}$$

where the right hand side of this relation is well–defined for $u_t \in V$ and φ, $\varphi_t \in V$.

We assume that V, W, X have a common subspace G – not necessarily closed. If $W \hookrightarrow X \hookrightarrow V$, as is commonly the case in applications, one takes $G = W$. We now define the principal set

$$K = \{\varphi : J \to G \mid \varphi \in C(J \to W) \cap C(J \to X) \cap C^1(J \to V)\}$$

and say that $u \in K$ is a (*strong*) *solution* of (2.1) if,

(a) $u_t(t) \in Y$ for a.a. $t \in J$, and $\langle Q(\cdot, u_t(\cdot)), \varphi(\cdot) \rangle_X : J \to \mathbb{R} \cup \{-\infty, \infty\}$ is measurable for all $\varphi \in K$;

(b) Distribution Identity:

$$\begin{aligned}\langle P(u_t(\tau)), \varphi(\tau) \rangle_V \Big|_0^t = \int_0^t \langle \{ P(u_t(\tau)), \varphi_t(\tau) \rangle_V &- \langle Q(\tau, u_t(\tau)), \varphi(\tau) \rangle_X \\ &- \langle A(u(\tau)), \varphi(\tau) \rangle_W + \langle F(u(\tau)), \varphi(\tau) \rangle_X \} d\tau\end{aligned}$$

for all $t \in J$ and $\varphi \in K$.

A third, and crucial, element in the definition of a solution of (2.1) is an appropriately formulated *conservation law* for the *energy* of a solution. To this end, the first task is to set up an appropriate energy functional. Proceeding formally, we write

$$\frac{d}{dt}\mathcal{A}(u(t)) = \langle A(u(t)), u_t(t)\rangle_W, \quad \frac{d}{dt}\mathcal{F}(u(t)) = \langle F(u(t)), u_t(t)\rangle_X$$

by virtue of the concept of Fréchet derivative. For $v \in V$ we introduce the *Hamiltonian* of the potential $\mathcal{P}$ by the formula

$$\mathcal{P}^*(v) = \langle P(v), v\rangle_V - \mathcal{P}(v) \tag{2.2}$$

(for Example 2 it is easy to see that $\mathcal{P}^*(v) = \mathcal{P}(v) = \frac{1}{2}\langle Pv, v\rangle_V$). Then, again formally,

$$\frac{d}{dt}\mathcal{P}^*(u_t(t)) = \langle \frac{d}{dt}P(u_t(t)), u_t(t)\rangle_V = \frac{d}{dt}\langle P(u_t(t)), u_t(t)\rangle_V - \langle P(u_t(t)), u_{tt}(t)\rangle_V;$$

this can be checked by assuming that $P(u_t)$ and u_t are both differentiable functions of t. Now put $\varphi = u_t$, formally, into the distribution identity (b); with the help of the above formal relations, we then get

$$\int_0^t \left\{ -\frac{d}{dt}\mathcal{P}^*(u_t(\tau)) - \frac{d}{dt}\mathcal{A}(u(\tau)) + \frac{d}{dt}\mathcal{F}(u(\tau)) - \langle Q(\tau, u_t(\tau)), u_t(\tau)\rangle_X \right\} d\tau = 0,$$

that is,

$$\mathcal{P}^*(u_t(\tau)) + \mathcal{A}(u(\tau)) - \mathcal{F}(u(\tau))\Big|_0^t = -\int_0^t \langle Q(\tau, u_t(\tau)), u_t(\tau)\rangle_X d\tau.$$

Since $u_t \notin X$, in general, the right hand side of the last relation is of course only formal. We now take the major step of introducing the following *third part* to the definition of solution of (2.1):

(c) Conservation Law: Let

$$\mathcal{E}u(t) = \mathcal{P}^*(u_t(t)) + \mathcal{A}(u(t)) - \mathcal{F}(u(t)) \tag{2.3}$$

be the *total energy of* u. There exists a function $\mathcal{D} : J \times Y \to [0, \infty]$, called the *dissipation rate*, such that $\mathcal{D}(\cdot, u_t(\cdot)) : J \to [0, \infty]$ is measurable and

$$\mathcal{E}u(t) - \mathcal{E}u(0) \le -\int_0^t \mathcal{D}(\tau, u_t(\tau))d\tau, \qquad t \in J. \tag{2.4}$$

Clearly $\mathcal{D}(\cdot, u_t(\cdot))$ is then locally integrable on J.

Condition (c) in this form first appears in [9, 17] – see particularly the discussion in [17, Section 2], and the related papers [12] and [23] concerning the existence of (strong) solutions of damped wave systems.

One must add to (c) a minimal relation between the dissipation rate $\mathcal{D}$ and the damping norm $\|Q(t,v)\|_{X'}$:

(C) *There is an exponent* $m>1$ *and a positive locally integrable function* $\delta=\delta(t)$ *such that*

$$\|Q(t,v)\|_{X'} \le [\delta(t)]^{1/m}[\mathcal{D}(t,v)]^{1/m'} \quad \text{for all } (t,v)\in J\times Y,$$

where m' *denotes the Hőlder conjugate of* m.

Condition (C) allows us to prove (what has so far not been done) that the term $\langle Q(\cdot,u_t(\cdot)),\varphi(\cdot)\rangle_X$ in the distribution identity (b) is locally integrable on J – note that it is measurable on J by (a). Indeed, for a.a. $t\in J$ we have

$$|\langle Q(t,u_t(t)),\varphi(t)\rangle_X| \le \|Q(t,u_t(t))\|_{X'}\|\varphi(t)\|_X \le [\delta(t)]^{1/m}[\mathcal{D}(t,u_t(t))]^{1/m'}\|\varphi(t)\|_X;$$

so, using Hőlder's inequality,

$$\int_0^t |\langle Q(\tau,u_t(\tau)),\varphi(\tau)\rangle_X|d\tau \le \left(\int_0^t \delta(\tau)d\tau\right)^{1/m}\left(\int_0^t \mathcal{D}(\tau,u_t(\tau))d\tau\right)^{1/m'}\sup_{[0,t]}\|\varphi(\tau)\|_X.$$

But this is locally finite, as required, since $\varphi\in C(J\to X)$ and $\mathcal{D}(\cdot,u_t(\cdot))$ is locally integrable on J.

Since the damping term Q is of crucial importance, we show how condition (C) arises from the natural choice

$$\mathcal{D}(t,v)=\langle Q(t,v),v\rangle_Y$$

in the case when

$$Q: J\times Y \to Y', \qquad X\hookrightarrow Y \quad \text{continuously},$$

and when, for all $t\in J$ and $v\in Y$,

$$\|Q(t,v)\|_{Y'} \le \hat{\delta}(t)\|v\|_Y^{m-1} \tag{i}$$

$$\|Q(t,v)\|_{Y'}\|v\|_Y \le \hat{\gamma}(t)\langle Q(t,v),v\rangle_Y \tag{ii}$$

(reverse pairing inequality).

The proof goes as follows, in the nontrivial case $v\neq 0$:

$$\begin{aligned}\|Q(t,v)\|_{Y'} &= \|Q(t,v)\|_{Y'}^{1/m}\|Q(t,v)\|_{Y'}^{1/m'}\\ &\le \{\hat{\delta}(t)\|v\|_Y^{m-1}\}^{1/m}\{\hat{\gamma}(t)\langle Q(t,v),v\rangle_Y/\|v\|_Y\}^{1/m'}\\ &= \{[\hat{\gamma}(t)]^{m-1}\hat{\delta}(t)\}^{1/m}[\mathcal{D}(t,v)]^{1/m'},\end{aligned}$$

the second step following in view of (i) and (ii). Finally, since $X\hookrightarrow Y$ continuously, we have

$$\|Q(t,v)\|_{X'} \le d\|Q(t,v)\|_{Y'} \quad \text{for all } (t,v)\in J\times Y$$

where d is a positive constant. Hence (C) holds with $\delta(t) = d^m[\hat{\gamma}(t)]^{m-1}\hat{\delta}(t)$.

3 THE PROBLEM OF BLOW–UP

This is, more precisely, the problem of global non–continuation of solutions for all $t \in J = [0, \infty)$. In many cases, however, the two problems can be the same – see Levine & Serrin [9].

For simplicity we provide a detailed discussion of the blow–up problem in only two cases,

$$(Pu_t)_t = -A(u) + F(u), \qquad t \in J, \tag{3.1}$$

and

$$Q(t, u_t) = -A(u) + F(u), \qquad t \in J, \tag{3.2}$$

where P is a symmetric operator from a Hilbert space V into V' (see Example 2 in Section 2, and recall that P must be linear and continuous), which moreover is assumed to be non–negative definite, namely $\mathcal{P} \geq 0$ on V. Assume also that A, F and Q are abstract operators on appropriate Banach spaces, as in the previous section.

Case (3.1) has no dissipation present, namely $Q \equiv 0$, while case (3.2) is the *parabolic* analogue of the main evolution equation (2.1). For both (3.1) and (3.2) the relation $Y \hookrightarrow V$, which was introduced at the beginning of the previous section in connection with the definition of the operator Q, is no longer needed since these spaces do not appear together.

CASE (3.1). We adjoin the energy relation (c). Since $Q \equiv 0$, and so $\mathcal{D} \equiv 0$, this relation takes the form

$$\mathcal{E}u(t) = \tfrac{1}{2}\langle Pu_t, u_t\rangle + \mathcal{A}(u(t)) - \mathcal{F}(u(t)) \leq \mathcal{E}u(0); \tag{3.3}$$

for simplicity in printing we have dropped the space subscript V from $\langle\cdot,\cdot\rangle$.

We now proceed following an idea of Levine [7], showing non–continuation whenever the initial energy is *negative*, that is $\mathcal{E}u(0) < 0$.

Thus let u be a solution on J. Define

$$\mathcal{I}(t) = \tfrac{1}{2}\langle Pu(t), u(t)\rangle + \beta(t),$$

where $\beta = \beta(t)$ is a twice differentiable function which we shall fix later in an appropriate way. Then since P is symmetric,

$$\mathcal{I}'(t) = \langle Pu(t), u_t(t)\rangle + \beta'(t),$$
$$\mathcal{I}''(t) = \langle Pu_t(t), u_t(t)\rangle - \langle A(u(t)), u(t)\rangle_W + \langle u(t), F(u(t))\rangle_X + \beta''(t),$$

by the main distribution identity with $\varphi = u \in K$.

Let us now introduce the following principal structure conditions on A and F:

$$q\mathcal{A}(u) \geq \langle A(u), u\rangle_W, \tag{A}$$

$$\langle F(u), u\rangle_X \geq q\mathcal{F}(u), \tag{D}$$

where $q > 0$. Hence

$$\begin{aligned} \mathcal{I}''(t) &\geq \langle Pu_t(t), u_t(t)\rangle - q\mathcal{A}(u(t)) + q\mathcal{F}(u(t)) + \beta''(t) \\ &\geq \left(1 + \tfrac{1}{2}q\right)\langle Pu_t(t), u_t(t)\rangle - q\mathcal{E}u(0) + \beta''(t), \end{aligned} \tag{3.4}$$

by virtue of the energy relation (3.3).

We now suppose that $\mathcal{E}u(0) < 0$, and choose

$$\beta(t) = \beta_0(t + t_0)^2,$$

with $\beta_0 = |\mathcal{E}u(0)|$ and $t_0 > 0$. Then

$$-q\mathcal{E}u(0) + \beta''(t) = (q + 2)|\mathcal{E}u(0)| = 4\beta_0(1 + \alpha),$$

where $\alpha = \frac{1}{4}(q - 2)$ – note that $q + 2 = 4(1 + \alpha)$.

At $t = 0$ we have

$$\mathcal{I}(0) = \tfrac{1}{2}\langle Pu(0), u(0)\rangle + \beta_0 t_0^2, \quad \mathcal{I}'(0) = \langle Pu(0), u_t(0)\rangle + 2\beta_0 t_0.$$

If t_0 is chosen sufficiently large, then $\mathcal{I}(0), \mathcal{I}'(0) > 0$. Moreover, of course, $\mathcal{I}''(t) > 0$ so that $\mathcal{I}'(t) > 0$, $\mathcal{I}(t) > 0$ for all $t \in J$.

Now set

$$\mathcal{J}(t) = [\mathcal{I}(t)]^{-\alpha}, \quad t \in J.$$

Then

$$\mathcal{J}'(t) = -\alpha[\mathcal{I}(t)]^{-(\alpha+1)}\mathcal{I}'(t)$$

$$\mathcal{J}''(t) = \alpha[\mathcal{I}(t)]^{-(\alpha+2)}\{(\alpha + 1)[\mathcal{I}'(t)]^2 - \mathcal{I}(t)\mathcal{I}''(t)\}.$$

LEMMA. *We have*

$$\mathcal{K}(t) = \mathcal{I}(t)\mathcal{I}''(t) - (\alpha + 1)[\mathcal{I}'(t)]^2 \geq 0, \quad t \in J.$$

Proof: By the earlier calculations, see in particular (3.4), together with the symmetry of P,

$$\begin{aligned} \mathcal{K}(t) &\geq \left\{\tfrac{1}{2}\langle Pu(t), u(t)\rangle + \beta(t)\right\}\{2(1 + \alpha)\langle Pu_t(t), u_t(t)\rangle + 4\beta_0(1 + \alpha)\} \\ &\qquad - (\alpha + 1)\left\{\langle Pu(t), u_t(t)\rangle + \beta'(t)\right\}^2 \\ &= (\alpha + 1)\{4\mathcal{P}(u(t))\mathcal{P}(u_t(t)) - |\langle Pu(t), u_t(t)\rangle|^2 \\ &\qquad + \beta_0\mathcal{P}(u(t) - (t + t_0)u_t(t))\} \geq 0 \end{aligned}$$

by the Cauchy–Schwarz inequality and the condition $\mathcal{P} \geq 0$ on V.

Assume $q > 2$. Then $\alpha > 0$. In turn

$$\mathcal{J}''(t) \leq 0, \quad \mathcal{J}(0) > 0, \quad \mathcal{J}'(0) < 0,$$

and therefore $\mathcal{J}$ reaches zero at a finite time t. But this is impossible, since $\mathcal{J}(t) > 0$ for all $t \in J$. This gives the following non–continuation result.

THEOREM 1. *Let* (A), (D) *be satisfied, with* $q > 2$. *Then no solution* u *of* (3.1) *can exist on* J *when* $\mathcal{E}u(0) < 0$.

EXAMPLE 1. If $A = -\Delta$ and $W = H_0^1(\Omega)$ in (3.1), then

$$\langle A(u(t)), u(t)\rangle_W = \int_\Omega |Du(t,x)|^2 dx = 2\mathcal{A}(u(t)),$$

so that the structure condition (A) is verified for any $q \geq 2$. When $Au = -\Delta_s u = -\text{div}(|\nabla u|^{s-2}\nabla u)$, $s > 1$, and $W = W_0^{1,s}(\Omega)$, we have

$$\langle A(u(t)), u(t)\rangle_W = \int_\Omega |Du(t,x)|^s dx = s\mathcal{A}(u(t)),$$

so that condition (A) is satisfied whenever $q \geq s$.

Thus to guarantee blow–up in these cases it is enough that F satisfies (D) with

$$q > 2 \quad \text{if } s \leq 2, \qquad q = s \quad \text{if } s > 2.$$

It should be noted particularly that in this example the domain Ω can be allowed to be unbounded.

CASE (3.2). For the equation (3.2) the solution set K should be slightly modified, by replacing the space $C^1(J \to V)$ by $AC(J \to Y)$, where AC denotes *absolutely continuous*, that is represented by the integral of an $L^1_{\text{loc}}(J)$ function.

Since $P = 0$, the energy conservation law (c) takes the form

$$\mathcal{E}u(t) \equiv \mathcal{A}(u(t)) - \mathcal{F}(u(t)) \leq \mathcal{E}u(0) - \int_0^t \mathcal{D}(\tau, u_t(\tau))d\tau. \tag{3.5}$$

We assume in addition to the earlier hypothesis (A) also the natural condition

$$\mathcal{A}(u) \geq 0 \quad \text{for any } u \in W \tag{A$'$}$$

and strengthen condition (D) to the form:

For every $\varepsilon > 0$ *there exist two positive constant* $c_1 = c_1(\varepsilon)$ *and* $c_2 = c_2(\varepsilon)$ *and an exponent* $p > 1$ *such that*

$$c_1\mathcal{F}(u) \leq c_2\|u\|_X^p \leq \langle F(u), u\rangle_X - q\mathcal{F}(u) \tag{D$'$}$$

for all $u \in G$ such that $\mathcal{F}(u) \geq \varepsilon$.

THEOREM 2 (see [10]). *Let* (A), (A)$'$, (D)$'$ *hold. Suppose that* $1 < m < p$, *where* m *is defined in* (C) *and*

$$\delta^{1/(1-m)} \notin L^1(J). \tag{3.6}$$

Then no solution u *of* (3.2) *can exist on* $J = [0, \infty)$ *when* $\mathcal{E}u(0) < 0$.

Proof: As in the previous demonstration we suppose the contrary, i.e., the existence of a solution on J. Consider the AC function

$$\mathcal{H}(t) = \int_0^t \mathcal{D}(\tau, u_t(\tau))d\tau - \mathcal{E}u(0).$$

Writing $-\mathcal{E}u(0) = \varepsilon > 0$, we get, since $\mathcal{A}$, $\mathcal{D} \geq 0$,

$$\mathcal{F}(u(t)) = \mathcal{A}(u(t)) - \mathcal{E}u(t) \geq -\mathcal{E}u(t) \geq \mathcal{H}(t) \geq \varepsilon > 0$$

on J. By the main distribution identity (b), with $P = 0$ and $\varphi = u \in K$, we obtain thanks to (A) and (D)$'$

$$\begin{aligned}
0 &= \langle F(u(t)), u(t)\rangle_X - \langle Q(t, u_t(t)), u(t)\rangle_X - \langle A(u(t)), u(t)\rangle_W \\
&\geq c_2\|u(t)\|_X^p + q\mathcal{F}(u(t)) - \langle Q(t, u_t(t)), u(t)\rangle_X - q\mathcal{A}(u(t)) \\
&= c_2\|u(t)\|_X^p - q\mathcal{E}u(t) - \langle Q(t, u_t(t)), u(t)\rangle_X.
\end{aligned}$$

Thus, recalling that $\mathcal{E}u(t) < 0$ on J, we see from the previous line that

$$\|Q(t, u_t(t))\|_{X'}\|u(t)\|_X \geq \langle Q(t, u_t(t)), u(t)\rangle_X > c_2\|u(t)\|_X^p.$$

This gives, since $p' = p/(p-1)$,

$$\begin{aligned}
\|Q(t, u_t(t))\|_{X'} &\geq c_2\|u(t)\|_X^{p-1} = c_2(\|u(t)\|_X^p)^{1/p'} \geq c_2\left(\frac{c_1}{c_2}\mathcal{F}(u(t))\right)^{1/p'} \\
&\geq C[\mathcal{H}(t)]^{1/p'},
\end{aligned}$$

where $C = c_1^{1/p'}c_2^{1/p}$. Note particularly that C depends on ε and thus on $\mathcal{E}u(0)$.

On the other hand, by (C),

$$0 < \|Q(t, u_t(t))\|_{X'} \leq [\delta(t)]^{1/m}[\mathcal{D}(t, u_t(t))]^{1/m'} = [\delta(t)]^{1/m}[\mathcal{H}'(t)]^{1/m'}.$$

Combining the last two lines yields

$$\mathcal{H}'(t) \geq [\delta(t)]^{-m'/m}\|Q(t, u_t(t))\|_{X'}^{m'} \geq C^{m'}[\delta(t)]^{1/(1-m)}[\mathcal{H}(t)]^{m'/p'}.$$

Now, since $1 < m < p$ by assumption,

$$\frac{m'}{p'} - 1 = m'\left(\frac{1}{p'} - \frac{1}{m'}\right) = m'\left(\frac{1}{m} - \frac{1}{p}\right) > 0,$$

so we can write $m'/p' = 1 + \vartheta$, $\vartheta > 0$. Then

$$\frac{\mathcal{H}'}{\mathcal{H}^{1+\vartheta}} \geq C^{m'} \delta^{1/(1-m)}$$

and by integration, setting $\mathcal{H}_0 = \mathcal{H}(0) = \varepsilon$,

$$\frac{1}{\vartheta \mathcal{H}_0^{\vartheta}} \geq \frac{1}{\vartheta [\mathcal{H}(t)]^{\vartheta}} + C^{m'} \int_0^t [\delta(\tau)]^{1/(1-m)} d\tau.$$

This is impossible, since the left hand side is finite and the right hand side goes to ∞ as $t \to \infty$.

If we take $\delta(t) = (1+t)^{\beta}$, then

$$[\delta(t)]^{1/(1-m)} = (1+t)^{\beta/(1-m)}$$

and the divergence condition is exactly the request that $\beta \leq m - 1$. It should be noted that this range includes $\beta = 0$, i.e. $\delta \equiv 1$.

COROLLARY. *Suppose* $\int_0^{\infty} [\delta(\tau)]^{1/(1-m)} d\tau = I$. *Then no global solution of* (3.2) *can exist if* $\mathcal{E}u(0) < 0$ *and, even more,*

$$\mathcal{E}u(0) = -\mathcal{H}_0 < -(\vartheta C^{m'} I)^{-1/\vartheta}, \qquad \vartheta = \frac{p-m}{(m-1)p}. \tag{3.7}$$

Note that (3.7) trivially holds whenever $\mathcal{E}u(0) < 0$ and $I = \infty$.

The operator Q can be allowed to depend on u provided that (C) is replaced by the following condition:

There are two exponents $m > 1$, $\kappa > 0$ *and a positive locally integrable function* $\delta = \delta(t)$ *such that*

$$\|Q(t,u,v)\|_{X'} \leq [\delta(t) \cdot \|u\|_X^{\kappa}]^{1/m} [\mathcal{D}(t,u,v)]^{1/m'} \quad \textit{for all } (t,u,v) \in J \times X \times Y.$$

Then the same results hold if we suppose

$$p > m + \kappa.$$

In this case

$$\vartheta = \frac{p - m - \kappa}{(m-1)p}.$$

EXAMPLE 2. The previous theorem can be applied to show blow–up for the degenerate parabolic equation

$$\delta(t)|u_t|^{m-2} u_t = a \operatorname{div}(|Du|^{s-2} Du) + c|u|^{p-2} u, \quad (t,x) \in J \times \Omega, \quad \Omega \subset \mathbb{R}^n,$$

where δ is a positive locally integrable function satisfying (3.6), a, $c > 0$, $1 < m < p$, $s > 1$, and Ω bounded.

In particular, we take $W = W_0^{1,s}(\Omega)$ and $X = L^p(\Omega)$. Then conditions (A), (A)$'$ hold with $q = s$, while (D)$'$ becomes

$$\frac{cc_1}{p}\|u\|_X^p \le c_2\|u\|_X^p \le c\left(\|u\|_X^p - \frac{s}{p}\|u\|_X^p\right).$$

Thus we must also have $1 < s < p$, and can take

$$c_2 = c\left(1 - \frac{s}{p}\right) > 0, \qquad c_1 = \frac{pc_2}{c} = p - s > 0.$$

Of course

$$Q(t,v) = \delta(t)|v|^{m-2}v, \qquad Q : J \times Y \to Y',$$

where $Y = L^m(\Omega)$. Then, taking $\mathcal{D}(t,y) = \delta(t)\|y\|_{L^m(\Omega)}^m$, it is easy to check that the main condition (C) is verified, for example by proceeding as in the discussion at the end of the previous section. Indeed (i) and (ii) hold with $\hat{\delta} = \delta$ and $\hat{\gamma} = 1$, while of course $X \hookrightarrow Y$ by the hypothesis $1 < m < p$ and the fact that Ω is bounded.

EXAMPLE 3. A second important case of (3.2) occurs when $Q(t,v) = Qv$, where Q is a linear continuous operator from Y to Y', and $X \hookrightarrow Y$. Then condition (C) holds with $\delta(t) =$ Constant and $\mathcal{D}(t,v) = \langle Qv, v\rangle_Y$ provided Q satisfies the additional requirement

$$\langle Qv, v\rangle_Y \ge \text{Pos. Const.}\, \|v\|_Y^2 \quad \text{for all } v \in Y.$$

(See also [6] when $A : W \to W'$ is linear, $Q : Y \to Y'$ is symmetric, and W, Y are real Hilbert spaces.)

Finally, we state without proof the corresponding non–continuation theorem for the abstract equation (2.1); see [9, Theorem 1].

THEOREM 3. *Assume that the previous conditions* (A), (A)$'$, (C), (D)$'$ *hold, that* $\mathcal{P}^* \ge 0$ *in* V, *and that* $X \hookrightarrow V$ *continuously. Suppose furthermore that there are constants* $\ell > 1$, $c_3 > 0$ *such that for all* $v \in V$

$$c_3\|P(v)\|_{V'}^{\ell'} \le (q+1)\langle P(v), v\rangle_V - q\mathcal{P}(v), \tag{E}$$

where q *is the exponent in* (A) *and* (D)$'$. *Assume finally that*

$$1 < \ell < p, \quad 1 < m < p \tag{3.8}$$

and

$$\int_0^\infty \frac{\min\{1, \delta^{(1+\vartheta)/(m-1)}\}}{\delta^{1/(m-1)}}\,dt = \infty \tag{3.9}$$

for some ϑ such that

$$0 < \vartheta < \min\left\{\frac{p-\ell}{p\ell - p + \ell},\ \frac{p-m}{pm-p+m}\right\}.$$

Then no solution u on J of (2.1) *can exist with $\mathcal{E}u(0) < 0$.*

In case $\delta(t) = (1+t)^\beta$ the divergence condition (3.9) is satisfied provided that $\beta \le m-1$.

EXAMPLE 4. Consider the degenerate wave equation with dissipation

$$u_{tt} + b|u_t|^{m-2}u_t = \Delta_s u + c|u|^{p-2}u, \quad (t,x) \in J \times \Omega, \quad \Omega \subset \mathbb{R}^n.$$

Here we suppose Ω bounded, m, p, $s > 1$, and take $P = I$, $V = L^2(\Omega)$, $W = W_0^{1,s}(\Omega)$, $X = L^p(\Omega)$; then $q = s$ for (A) and $\ell = 2$, $c_3 = 1 + q/2$ for (E), see Example 2 of Section 2.

As in Example 2 above, for (D)$'$ we need $p > s$. Recalling the hypothesis (3.8) then gives the principal exponent condition

$$p > \max\{2,\ m,\ s\}. \tag{3.10}$$

Of course $Q(t,v) = Q(v) = b|v|^{m-2}v$; the choice of an appropriate space Y is here complicated by the condition $Y \hookrightarrow V$ for the definition of Q and the requiremnt that $Q : Y \to X'$. A little reflection shows that the required space Y is given by

$$Y = \begin{cases} L^2(\Omega), & \text{if } 1 < m < 2 \\ L^m(\Omega), & \text{if } \quad m \ge 2. \end{cases}$$

Then it is easy to check that (C) holds with $\mathcal{D}(t,v) = \mathcal{D}(v) = b\|v\|^m_{L^m(\Omega)}$ and $\delta(t) = b|\Omega|^{(p-m)/m}$.

In [4] the special case $p > m > 2$ was obtained when $s = 2$, that is $\Delta_s = \Delta$.

Generalizations to time dependent potentials $\mathcal{A}$ and $\mathcal{F}$ will appear in work of Levine, Pucci & Serrin [11].

4 STABILITY

Here we are interested in the converse of the blow–up problem, that is, under which conditions does $\mathcal{E}u(t) \to 0$ as $t \to \infty$? We shall concentrate on perhaps the most interesting situation – in which stability holds when the initial data is small, and blow–up when the data is large, i.e. the case of the previous section. We shall treat the general equation (2.1), our discussion being based on work of Pucci & Serrin [18, 20] and Boccuto & Vitillaro [3].

Naturally, one cannot expect $\mathcal{E}u(t) \to 0$ as $t \to \infty$ unless the damping term Q is sufficiently strong. Thus for stability we shall require the following further condition on the dissipation function $\mathcal{D}$.

(C)′ *There is a non–negative function σ on J, and a wedge function ω, i.e. a non–decreasing function on $[0,\infty)$ with $\omega(0)=0$, $\omega(\tau)>0$ if $\tau \neq 0$, such that*

$$\omega(\|v\|_Y) \le \sigma(t)\mathcal{D}(t,v) \quad \text{on } J \times Y.$$

We assume the *existence of numbers $a_0 > 0$, $s > 1$ such that*

$$\langle A(u), u\rangle_W \ge a_0\|u\|_W^s \quad \text{for } u \in G, \tag{A)''$$

and we suppose that *there exist constants $p > s$, $\mu \in [0, a_0)$, $c > 0$ such that*

$$\langle F(u), u\rangle_X \le \mu\|u\|_W^s + c\|u\|_X^p \quad \text{for } u \in G. \tag{D)''$$

It will be assumed that $\mathcal{P}^* \ge 0$ on V, while instead of (E) we now suppose for stability that

(E)′ *For some $d > 0$ the sets $P(E)$ and $\mathcal{P}(E)$ are bounded in V' and $\mathbb{R}$, respectively, where $E = E_d = \{v \in V \, : \, \mathcal{P}^*(v) \le d\}$.*

Note that (A)″ implies $s\mathcal{A}(u) \ge a_0\|u\|_W^s$ on G, which is related to (A) and stronger than (A)′. Condition (E)′ can be replaced by the simpler but stronger requirement

$$\mathcal{P}^*(v) \to 0 \quad \textit{implies} \quad \|v\|_V \to 0.$$

We shall finally require the continuous embeddings

$$W \hookrightarrow X \hookrightarrow V.$$

Thus we can take $G = W$; moreover there is $C > 0$ such that $\|u\|_X \le C\|u\|_W$ for all $u \in G$.

We can now state the following main result of local asymptotic stability.

THEOREM ([18, 20, 3]). *Let* (A)″, (C), (C)′, (D)″, (E)′ *hold, and suppose also that*

$$\liminf_{t\to\infty} \frac{1}{t^m} \int_0^t \{\delta(\tau) + [\sigma(\tau)]^{m-1}\} d\tau < \infty.$$

Let u be a solution of (2.1) *such that*

$$\mathcal{E}u : J \to \mathbb{R} \quad \text{is non–increasing on } J. \tag{4.1}$$

Then

$$\mathcal{E}u(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{4.2}$$

provided that the initial data $u(0)$, $u_t(0)$ has sufficiently small norms in W and V, respectively.

Note that (4.1) does *not* follow from (2.4), but *would* hold if in (2.4) the *equality* is enforced.

The particular case $\delta(t) = (1+t)^\beta$, $\sigma(t) = (1+t)^\gamma$ yields the conditions

$$\beta \le m-1, \qquad \gamma \le 1.$$

Since generally $\sigma\delta \ge 1$ we expect $-1 \le -\gamma \le \beta \le m-1$. If δ, σ are constant (i.e. autonomous damping) these conditions are satisfied with $\beta = \gamma = 0$.

Proof of the theorem: Our first goal is to obtain an estimate for the required smallness of the initial data, after which the actual limit of the energy must be established. In view of the previous blow–up result this first part is crucial – since large data lead to blow–up. We divide the proof into several steps.

LEMMA 1. *Under the hypotheses above, for any solution u of* (2.1) *we have, for every $t \in J$,*

$$\mathcal{E}u(t) \ge \mathcal{P}^*(u_t(t)) + 2a\|u(t)\|_X^s - \frac{c}{p}\|u(t)\|_X^p, \qquad \text{where } a = \frac{a_0-\mu}{2sC^s}.$$

Proof: From (2.3) we get by (A)$''$ and (D)$''$

$$\begin{aligned}\mathcal{E}u(t) &= \mathcal{P}^*(u_t(t)) + \mathcal{A}(u(t)) - \mathcal{F}(u(t)) \\ &\ge \mathcal{P}^*(u_t(t)) + \frac{a_0}{s}\|u(t)\|_W^s - \frac{\mu}{s}\|u(t)\|_W^s - \frac{c}{p}\|u(t)\|_X^p \\ &= \mathcal{P}^*(u_t(t)) + \frac{a_0-\mu}{s}\|u(t)\|_W^s - \frac{c}{p}\|u(t)\|_X^p, \end{aligned} \tag{4.3}$$

which gives the result since $\|u(t)\|_W \ge \|u(t)\|_X/C$.

Let

$$\Sigma_0 = \{(\lambda,\mathcal{E}) \in \mathbb{R}^2 \ : \ 0 \le \lambda < \lambda_0,\ 2a\lambda^s - \frac{c}{p}\lambda^p \le \mathcal{E} < \mathcal{E}_0\},$$

where

$$\lambda_0 = \left(\frac{2as}{c}\right)^{1/(p-s)}, \qquad \mathcal{E}_0 = 2a\lambda_0^s\left(1-\frac{s}{p}\right).$$

By the lemma, if the initial data $u(0)$, $u_t(0)$ is such that the point $(\|u(0)\|_X, \mathcal{E}u(0))$ is in Σ_0, then, because $\mathcal{E}u$ is non–increasing on J by (4.1) and $\mathcal{P}^* \ge 0$ on V, it is easy to see that *for all $t \in J$* the point $(\|u(t)\|_X, \mathcal{E}u(t))$ remains in Σ_0. Thus Σ_0 is a *potential well* for the problem.

It is worth noting that one cannot have $\mathcal{E}u(0) < 0$ unless the data is fairly large, e.g., by Lemma 1,

$$\|u(0)\|_X \ge \left(\frac{2ap}{c}\right)^{1/(p-s)} = \lambda_2;$$

clearly, $\lambda_2 > \lambda_0$ since $p > s$ by (D)$''$.

The next problem is to ensure that $\mathcal{E}u(t) \to 0$ as $t \to \infty$. To this end, it is necessary that the initial point $(\|u(0)\|_X, \mathcal{E}u(0))$ lies in the *smaller set* Σ defined by

$$\Sigma = \{(\lambda, \mathcal{E}) \in \mathbb{R}^2 \ : \ 0 \le \lambda < \lambda_1,\ \ 2a\lambda^s - \frac{c}{p}\lambda^p \le \mathcal{E} < \mathcal{E}_1\},$$

where

$$\lambda_1 = \left(\frac{a}{c}\right)^{1/(p-s)}, \qquad \mathcal{E}_1 = \min\left\{d,\ a\lambda_1^s\left(2 - \frac{1}{p}\right)\right\}.$$

As above, once the *phase trajectory* $(\|u(t)\|_X, \mathcal{E}u(t))$ enters Σ it remains there for all larger $t \in J$. This remark is crucial in the following proofs.

LEMMA 2. *Let* $(\|u(0)\|_X, \mathcal{E}u(0)) \in \Sigma$. *Then, under the hypotheses of the theorem,*

$$\mathcal{E}u(t) \ge \mathcal{P}^*(u_t(t)) + \frac{a_0 - \mu}{2s}\|u\|_W^s \quad \textit{in } J.$$

Proof: Clearly

$$a\lambda^s - \frac{c}{p}\lambda^p \ge a\lambda^s\left(1 - \frac{1}{p}\right) \ge 0 \quad \text{whenever } \lambda \le \lambda_1. \tag{4.4}$$

Also

$$\frac{a_0 - \mu}{s}\|u(t)\|_W^s \ge \frac{a_0 - \mu}{2s}\|u(t)\|_W^s + a\|u(t)\|_X^s, \tag{4.5}$$

so the result follows from (4.3), (4.5) and (4.4).

LEMMA 3. *Let* $(\|u(0)\|_X, \mathcal{E}u(0)) \in \Sigma$. *Then, under the hypotheses of the theorem,* (4.2) *holds.*

Proof: We outline the ideas involved. Assume for contradiction that (4.2) fails. Then by (4.1) there exists $l > 0$ such that

$$\mathcal{E}u(t) \ge l \quad \text{for all } t \in J.$$

By the distribution identity (b) with $\varphi = u \in K$, together with the definition (2.2) of $\mathcal{P}^*$, we obtain

$$\begin{aligned} \frac{d}{dt}\langle P(u_t(t)), u(t)\rangle_V = \{&\mathcal{P}(u_t(t)) + 2\mathcal{P}^*(u_t(t))\} \\ &- \{\mathcal{P}^*(u_t(t)) + \langle A(u(t)), u(t)\rangle_W - \langle F(u(t)), u(t)\rangle_X\} \\ &- \langle Q(t, u_t(t)), u(t)\rangle_X. \end{aligned} \tag{4.6}$$

Next we assert that for any $\theta > 0$ there is $\gamma(\theta) > 0$ such that for all $t \ge T > 0$

$$\int_T^t \{\mathcal{P}(u_t(\tau)) + 2\mathcal{P}^*(u_t(\tau))\}d\tau \le \theta t + \gamma(\theta)\varepsilon(T)\left(\int_0^t [\sigma(\tau)]^{m-1}d\tau\right)^{1/m} \tag{4.7}$$

$$\left|\int_T^t \langle Q(\tau, u_t(\tau)), u(\tau)\rangle_X d\tau\right| \le \varepsilon(T)\left(\int_0^t \delta(\tau)d\tau\right)^{1/m}, \tag{4.8}$$

where $\varepsilon(T) \to 0$ as $t \to \infty$. To obtain (4.7), note by Lemma 2 and (4.1) that

$$0 \le \mathcal{P}^*(u_t(t)) \le \mathcal{E}u(t) \le \mathcal{E}u(0) < \mathcal{E}_1 \le d, \quad t \in J,$$

so by (E)$'$ also $\mathcal{P}(u_t)$ is bounded on J. Moreover $\mathcal{D}(\cdot, u_t(\cdot)) \in L^1(J)$ by the conservation law (c) and the fact that $\mathcal{E}u$ is bounded on J. The result then follows with the help of (C)$'$ and the fact that both $\mathcal{P}$ and $\mathcal{P}^*$ are continuous and vanish at $v = 0$; see [18, inequality (3.7)].

Next, (4.8) is an easy consequence of (C), and the facts that $\mathcal{D}(\cdot, u_t(\cdot)) \in L^1(J)$ and $\|u(t)\|_X < \lambda_1$ on J; see [18, inequality (3.10)].

Furthermore, from Lemma 2, the fact that $\mathcal{P}^* \ge 0$ on V, and the continuity of $\mathcal{A} : W \to \mathbb{R}$ and $\mathcal{F} : X \to \mathbb{R}$, it can be shown (see [3, Lemma 4.6; 20]) that for all $l > 0$ there is $\alpha(l) > 0$ such that

$$\mathcal{P}^*(u_t(t)) + \langle A(u(t)), u(t)\rangle_W - \langle F(u(t)), u(t)\rangle_X \ge \alpha(l) \tag{4.9}$$

for all $t \ge T$.

It now follows from (4.6)–(4.9) that for $t \ge T$

$$\langle P(u_t(\tau)), u(\tau)\rangle_V \Big|_T^t \le [\theta - \alpha(l)]t + \varepsilon(T)\left\{\gamma(\theta)\left(\int_0^t [\sigma(\tau)]^{m-1} d\tau\right)^{1/m} + \left(\int_0^t \delta(\tau) d\tau\right)^{1/m}\right\}.$$

Choose a sequence $(t_i) \nearrow \infty$ such that, for an appropriate constant M,

$$\frac{1}{t_i^m}\int_0^{t_i} \{\delta(\tau) + [\sigma(\tau)]^{m-1}\} d\tau \le M^m.$$

Consequently for all $t_i \ge T$

$$\langle P(u_t(\tau)), u(\tau)\rangle_V \Big|_T^{t_i} \le \{\theta - \alpha(l) + M\varepsilon(T)[1 + \gamma(\theta)]\}t_i.$$

Take $\theta = \alpha(l)/4$ and T so large that

$$M[1 + \gamma(\alpha(l)/4)]\varepsilon(T) \le \alpha(l)/4.$$

Hence

$$\langle P(u_t(\tau)), u(\tau)\rangle_V \Big|_T^{t_i} \le -\tfrac{1}{2}\alpha(l)t_i. \tag{4.10}$$

We claim that $\|u(\cdot)\|_V$ and $\|P(u_t(\cdot))\|_{V'}$ are bounded on J, when $(\|u(0)\|_X, \mathcal{E}u(0))$ is in Σ. Indeed, see above,

$$0 \le \mathcal{P}^*(u_t(t)) < d \quad \text{on } J.$$

Hence by (E)$'$ it follows that $\|P(u_t(\cdot))\|_{V'}$ is bounded on J.

Moreover, by the embedding $X \hookrightarrow V$ we get

$$\|u(t)\|_V \le \text{Const.}\,\|u(t)\|_X \le \text{Const.}\,\lambda_1 \qquad \text{on } J.$$

This completes the proof of the claim. In turn, we reach a contradiction with (4.10) when $t_i \to \infty$.

This contradiction concludes the proof of the theorem provided we show that $(\|u(0)\|_X, \mathcal{E}u(0))$ is in Σ whenever $\|u(0)\|_W$ and $\|u_t(0)\|_V$ are sufficiently small. This is, however, a consequence of the fact that $\|u(0)\|_X \le C\|u(0)\|_W$, the definition (2.3) and the fact that the potentials $\mathcal{A}$ and $\mathcal{F}$, and the Hamiltonian $\mathcal{P}^*$ are continuous and normalized so that $\mathcal{A}(0) = 0$, $\mathcal{F}(0) = 0$, $\mathcal{P}^*(0) = 0$ in W, X and V, respectively.

EXAMPLE. Consider the degenerate wave equation

$$u_{tt} - \Delta_s u + \tilde{Q}(t, x, u_t) = f(x, u), \quad (t, x) \in J \times \Omega, \quad \Omega \subset \mathbb{R}^n.$$

Here Ω is bounded, $P = I$, $V = L^2(\Omega)$ and $s > 1$, $W = W_0^{1,s}(\Omega)$. For simplicity take $X = L^p(\Omega)$, $p \le r$, where r is the Sobolev exponent for $W_0^{1,s}(\Omega)$. The embedding $W \hookrightarrow X$ is then the Sobolev theorem and C is the Sobolev constant. We suppose that the Nemitsky operator Q corresponding to the damping term $\tilde{Q}$ verifies (C), (C)$'$, see Example 4 in Section 2.

The conditions (A), (A)$'$, (A)$''$ are easily checked, in particular with $q = s$ and $a_0 = 1$. Moreover, for the specific function

$$f(x, u) = \hat{\mu}|u|^{s-2}u + c|u|^{p-2}u, \quad s < p \le r, \quad c > 0,$$

the condition (D)$''$ is verified provided that $\hat{\mu} < \mu_0$. Here μ_0 is the Poincaré constant, that is the reciprocal of the *first eigenvalue* of $-\Delta_s$ in Ω with Dirichlet homogeneous boundary conditions, namely,

$$\|u\|_{L^s} \le \mu_0^{-1/s}\|u\|_{W_0^{1,s}},$$

and we can take $\mu = \hat{\mu}/\mu_0 < 1 = a_0$. To verify (D)$'$ when $q = s$ one takes $c_2 = c(1 - s/p)$ as in Example 2 of Section 3, and $c_1 = c_1(\varepsilon) > 0$ sufficiently small, depending on p, s, $\hat{\mu}$, c, $|\Omega|$ – see [11, Section 4] for a complete discussion.

Finally, (E) holds with $\ell = 2$, $c_3 = 1 + q/2$, while (E)$'$ is clear since in the present case $\|P(v)\|_{V'} = \|v\|_{L^2(\Omega)}$ and $\mathcal{P}^*(v) = \mathcal{P}(v) = \frac{1}{2}\|v\|^2_{L^2(\Omega)}$. Thus if $\mathcal{P}^*(v) \le d$ we get $\|P(v)\|_{V'} \le \sqrt{2d}$ and $\mathcal{P}(v) \le d$.

This example is of particular interest because *both* the blow–up Theorem 3 of Section 3 *as well as* the stability theorem above are applicable.

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