SIAM Math. Anal. **25** (1994), 815–835.

ASYMPTOTIC STABILITY FOR INTERMITTENTLY CONTROLLED NONLINEAR OSCILLATORS

Patrizia Pucci & James Serrin

Abstract. We prove a number of asymptotic stability theorems for intermittently damped quasi–variational systems, extending and generalizing various previous work on the subject.

Key words. Global asymptotic stability, intermittent damping, control set.

AMS(MOS) subject classifications. 34 D XX, 35 A 15.

§1. Introduction.

The problem of global asymptotic stability of solutions of second order equations with intermittent damping has been studied by Smith, Thurston & Wong, Artstein & Infante, Murakami and Hatvani & Totik. In this paper we give various generalizations and extensions of their work to quasi–variational systems.

As in our earlier work [5], [6] on asymptotic stability, we consider vector unknowns $u: J \to \mathbb{R}^N$ and systems having the general form

$$\big(\nabla\mathcal{L}(t,u,u')\big)' - \nabla_u\mathcal{L}(t,u,u') = Q(t,u,u'), \qquad t\in J, \tag{1.1}$$

where J is a half open interval of the form $[T,\infty)$ and $\mathcal{L}(t,u,p) = G(u,p) - F(t,u)$, and where G, F, Q are given continuously differentiable functions. The most important of the conditions which will be imposed on (1.1) are that

$$G(u,\cdot) \quad \text{is strictly convex in } \mathbb{R}^N; \qquad G(u,0)=0, \quad \nabla G(u,0)=0, \tag{1.2}$$

$$\big(\nabla_u F(t,u),u\big) > 0 \quad \text{for } u\neq 0; \qquad F(t,u)=0, \tag{1.3}$$

$$\big(Q(t,u,p),p\big) \le 0. \tag{1.4}$$

Here $(\cdot\,,\cdot)$ denotes the inner product in $\mathbb{R}^N$ and

$$\nabla = \nabla_p = \left(\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_N}\right), \qquad \nabla_u = \left(\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_N}\right).$$

The function F represents a restoring potential and Q a general nonlinear damping, expressed by (1.4). In Section 2 we shall give a complete set of hypotheses, while explicit examples are given in [5] and [6].

Since $\nabla G(u,0) = \nabla_u G(u,0) = \nabla_u F(t,0) = Q(t,u,0) = 0$ it is clear that the rest state $u = 0$ is a solution of (1.1). This state is said to be *a global attractor* for the system if any bounded solution u, defined on some interval J, has the property

$$u(t),\ u'(t) \to 0 \quad \text{as } t \to \infty.$$

By the concept of *intermittent damping* we mean that certain restrictions or controls are placed on the damping term on a sequence of non–overlapping intervals $I_n = [a_n, b_n]$ of J, with $a_n \to \infty$; on the other hand, in the gaps *between* these intervals either no restrictions are imposed or, alternatively, the damping is assumed to be bounded from zero but to be otherwise uncontrolled. We emphasize that the intervals I_n may be arbitrarily widely spaced, leaving gaps between them which can be as long as one wishes.

Our purpose is to show that under appropriate conditions on the measures $|I_n|$ and on the damping term $Q(t,u,p)$ for $t \in \bigcup_1^\infty I_n$, the rest state $u = 0$ becomes a global attractor for (1.1).

From a mechanical point of view the system (1.1) can be considered as the governing law of a holonomic dynamical system, having N degrees of freedom and subject to nonlinear damping. The notion of intermittent damping then occurs if the system is positively damped in the time intervals I_n, but has its damping either *switched off* or *unrestricted* at other times. The system is oscillatory when no damping is present, because $(f(t,u),u) > 0$ for $u \neq 0$; that is, it is not possible to have any solution, other than the trivial one $u = 0$, approaching a limit as $t \to \infty$ (see [5], Section 5, for a more complete discussion). From this point of view the question we consider is whether the damping which occurs on the time intervals I_n is sufficient to drive the solution to its rest state as $t \to \infty$. The following example provides a specific illustration of this situation in perhaps its simplest form.

Consider the system

$$u'' + A(t,u,u')u' + f(u) = 0, \tag{1.5}$$

where A is a continuous $N \times N$ non–negative definite matrix and $f(u) = \nabla F(u)$. This system arises from (1.1) by the specializations

$$G(p) = \tfrac{1}{2}|p|^2, \qquad Q(t,u,p) = -A(t,u,p)p.$$

We suppose that $(f(u),u) > 0$ for $u \neq 0$, and that A is bounded and uniformly positive definite for $t \in I = \bigcup_1^\infty I_n$ and (u,p) in any given compact set of $\mathbb{R}^N \times \mathbb{R}^N$; no restrictions, however, other than non–negativity, are placed on A in the set $J \setminus I$. Then the following rather unexpected result holds:

If the measures of the intervals I_n satisfy

$$\sum_{1}^{\infty} |I_n|^3 = \infty, \tag{1.6}$$

then $u = 0$ is a global attractor for (1.5).

The exponent 3 *is best possible: that is, without further restrictions no smaller exponent can yield the general conclusion.*

A stronger result is valid if the damping matrix A has the decomposition

$$A(t, u, p) = \beta(t, u, p)I + B(t, u, p), \tag{1.7}$$

where $B(t, u, p)$ is bounded and non–negative definite for $t \in J$ and (u, p) in any compact set of $\mathbb{R}^N \times \mathbb{R}^N$; the coefficient $\beta(t, u, p)$ is such that for every compact set K of $\mathbb{R}^N \times \mathbb{R}^N$ there exist positive constants β_1, β_2 such that

$$\beta(t, u, p) \geq \beta_1 \qquad \text{in } J \times K, \tag{1.8}$$

$$\beta(t, u, p) \leq \beta_2 \qquad \text{in } I \times K. \tag{1.9}$$

Then $u = 0$ is a global attractor for (1.5) *provided that*

$$\sum_{1}^{\infty} |I_n|^2 = \infty. \tag{1.10}$$

Again the exponent is best possible.

The above results are special cases respectively of Corollaries 3 and 4 in Section 3, see the comments at the end of Section 3. Indeed in those results the damping need not even be bounded on I but only have a controlled L^1 norm. Moreover the constant β_1 in (1.8) can be replaced by a non–negative measurable function $\hat{\sigma}$ satisfying a *positive mean value criterion*, see condition (2.13) below.

References [1–4], [7] and [8] treat the case $N = 1$ of (1.5); moreover in [7] the coefficient A is independent of u, u' and $f(u)$ is linear. Our results are improvements of the corresponding ones in these papers, even when restricted to the cases treated there.

In Section 2 we present the setting of the paper and state two important preliminary theorems upon which our further results are based. The main results for the system (1.1) are given in Section 3 and proved in Sections 4 and 5. In Section 6 we present specific examples showing that the exponents 2 and 3 in the above results are best possible.

§2. Preliminaries.

We consider vector solutions $u = (u_1, \ldots, u_N)$ of the quasi–variational ordinary differential system

$$\big(\nabla G(u, u')\big)' - \nabla_u G(u, u') + f(t, u) = Q(t, u, u'), \qquad t \in J = [T, \infty), \tag{2.1}$$

where ∇ denotes the gradient operator with respect to the variable p and

$$f(t, u) = \nabla_u F(t, u).$$

It will be supposed throughout the paper that

$$G \in C^1(\mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}), \quad F \in C^1(J \times \mathbb{R}^N; \mathbb{R}), \quad Q \in C(J \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^N),$$

and also that the following natural conditions hold:

(H_1) $G(u, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $u \in \mathbb{R}^N$; with $G(u, 0) = 0$ and $\nabla G(u, 0) = 0$. For all $U > 0$ there exists a positive constant $\Theta = \Theta(U)$ and an exponent $m > 1$ independent of U such that

$$|\nabla G(u, p)| \le \Theta |p|^{m-1} \quad \text{for all } |u| \le U \text{ and } |p| \le 1. \tag{2.2}$$

(H_2) $F(t, 0) = 0$ for all $t \in J$. For all u_0, U with $0 < u_0 \le U$ there exists a constant $\kappa > 0$ and a non–negative function $\psi \in L^1(J)$ such that

$$\big(f(t, u), u\big) \ge \kappa \qquad \text{when } t \in J \qquad \text{and} \qquad |u| \in [u_0, U], \tag{2.3}$$

$$|F_t(t, u)| \le \psi(t) \qquad \text{when } t \in J(\text{a.e.}) \quad \text{and} \qquad |u| \le U. \tag{2.4}$$

(H_3) $\big(Q(t, u, p), p\big) \le 0$ for all $t \in J$, $u \in \mathbb{R}^N$ and $p \in \mathbb{R}^N$.

If F does not depend on t, then (2.3) follows from the condition $\big(f(u), u\big) > 0$ for $u \ne 0$, while (2.4) is irrelevant. Finally, when $N = 1$ any function f is of gradient type, with $F(t, u) = \int_0^u f(t, s)\, ds$.

Obviously (H_1) is satisfied by any strictly convex homogeneous function $G = G(p)$ of degree $m > 1$, and in particular by the model function $G(p) = |p|^m/m$, $m > 1$; another example is $G(p) = \sqrt{1 + |p|^2} - 1$, with $m = 2$. The system (1.5) arises when $G(p) = \frac{1}{2}|p|^2$, with the corresponding exponent $m = 2$.

The next hypothesis places in evidence the concept of a control set $I \subset J$ where the damping term Q is subject to restrictions.

(H_4) For all $U > 0$ there exists a measurable control set $I \subset J$ and a number $\gamma \ge 1$ such that

$$|Q(t, u, p)| \cdot |p| \le \gamma \left|\big(Q(t, u, p), p\big)\right| \quad \text{for all } t \in I,\ |u| \le U \text{ and } p \in \mathbb{R}^N. \tag{2.5}$$

Moreover there exists a positive measurable damping function $\delta : I \to \mathbb{R}$ and numbers μ, $q > 0$ such that

$$(2.6) \qquad \big(Q(t,u,p),u\big) \le \delta(t)\,|p|^{\mu} \quad \text{for } t \in I,\ |u| \le U \text{ and } |p| \le q.$$

Although I, δ and γ, μ, q may depend on U, for simplicity we do not specifically indicate this dependence. When $N = 1$ condition (2.5) holds automatically with $\gamma = 1$.

In [5] we considered the asymptotic stability of the system (2.1) when the damping magnitude $|Q|$ is controlled from below, but not bounded away from zero. Specifically the following condition was required:

For every $U > 0$ there exist a non–negative measurable damping control $\sigma : I \to \mathbb{R}$ and an exponent $\nu > 0$ such that

$$(2.7) \qquad |Q(t,u,p)| \ge \sigma(t)\min\{1,\ |p|^{\nu}\} \quad \text{for all } t \in I,\ |u| \le U \text{ and } p \in \mathbb{R}^N.$$

A further technical hypothesis was also assumed concerning the function G:

For every $U > 0$ and $p_0 > 0$ there is a constant such that

$$(2.8) \qquad \big(\nabla_u G(u,p),u\big) \le \text{ Const. } \big(\nabla G(u,p),p\big) \quad \text{whenever } |u| \le U \text{ and } |p| \ge p_0.$$

Note that (2.8) holds whenever $G(u,p) = g(u)\overline{G}(p)$, with $g(u) > 0$ a smooth function in $\mathbb{R}^N$ and $\overline{G}$ satisfying (H_1).

Under the natural assumptions (H_1)–(H_4), together with (2.7) and (2.8), the following result is valid, see [5, Theorem 4.2] and the modified version of this result proved in Section 3.2 of [4]. This will be the basis for the first main theorem in Section 3. In its statement we agree that the function δk is extended to all of J by the definition $\delta(t)k(t) = 0$ for $t \in J \setminus I$.

THEOREM A. *Assume that for every $U > 0$ there exists a bounded absolutely continuous function k on J such that*

$$(2.9) \qquad k \notin L^1(J), \qquad k = 0 \quad \textit{on} \quad J \setminus I,$$

$$(2.10) \qquad 0 \le k \le \text{Const. } \sigma \ \textit{on } I, \qquad |k'| \le \text{Const. } \sigma^{\lambda}k^{1-\lambda} \quad \textit{a.e. on } I,$$

where

$$(2.11) \qquad \lambda = \begin{cases} \dfrac{m-1}{\nu+1}, & \textit{if } \nu > m-2 \\ 1, & \textit{if } \nu \le m-2 \quad (\textit{and } m > 2). \end{cases}$$

Suppose furthermore that there exists a constant $M > 0$ *for which*

$$\int_T^t \delta(s) k^{\mu+1}(s)\, ds \le M \int_T^t k(s)\, ds, \quad t \in J. \tag{2.12}$$

Then the rest state $u = 0$ *is a global attractor for the system* (2.1).

In [6] we also studied asymptotic stability for the complementary situation in which the damping magnitude $|Q|$ is bounded from zero when $|u|$ and $|p|$ are bounded from zero. In particular the following condition was assumed:

There is (i) a continuous function $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ with

$$\varphi(u,p) > 0 \qquad \text{when } u \neq 0 \text{ and } p \neq 0;$$

(ii) a measurable function $\hat\sigma : J \to [0,\infty)$ satisfying

$$\int_L \hat\sigma(t)\,dt \geq a(|L|) > 0 \quad \text{for all intervals } L \subset J \text{ with } |L| \in (0,1);$$

such that for all $U > 0$ there holds

$$\big|\big(Q(t,u,p),p\big)\big| \geq \hat\sigma(t)\,\varphi(u,p) \quad \text{for all } t \in J,\ |u| \leq U \text{ and } |p| \leq q, \tag{2.13}$$

where $q = q(U) > 0$ is given in (H_4).

This is in fact condition (H_3) of [6], in the weaker version involving (ii) which was given in Section 7 of [6]. (In this context, condition (ii) was first introduced by Hatvani [2].) If $Q(t,u,p) = \hat\sigma(t)\,\hat Q(u,p)$ and $\big(\hat Q(u,p),p\big) < 0$ for $u,p \neq 0$, it is easy to see that (2.13) is satisfied.

Two further technical hypotheses were introduced in [6]; they are required only when $N > 1$, though in fact the second, (V_2), automatically holds when $N = 1$ with $\varepsilon(p) = 0$, $g(u) = \frac{1}{2}u^2$ and $C = 0$ in view of (H_1) and (H_3); see also [6, Lemma 2.1].

(V_1) For all $U > 0$ and $p_0 > 0$ there is a non–negative measurable function $h \notin L^1(J)$ such that

$$\big|\big(Q(t,u,p),p\big)\big| \geq h(t) \quad \text{for all } t \in J,\ |u| \leq U \text{ and } |p| \geq p_0.$$

(V_2) For all $U > 0$ there exists a continuous function $\varepsilon(p)$ with $\varepsilon(0) = 0$, such that

$$\big(Q(t,u,p),u\big) \leq \varepsilon(p)$$

when $t \in J$, $|u| \leq U$, $|p| \leq q$ and $\big(\nabla G(u,p),u\big) \geq 0$. Moreover there exists a C^1 function $g(u)$ and a constant $C \geq 0$ such that

$$\frac{\big(\nabla_u g(u),p\big)}{|p|} - \frac{\big(\nabla G(u,p),u\big)}{|\nabla G(u,p)|} \leq C\,\frac{\varphi(u,p)}{|p|},$$

when $|u| \leq U$, $|p| \leq q$ and $(\nabla G(u,p),u) < 0$. Again q and φ are given in (2.13).

It is worth noting that (V_2) is satisfied if the vectors p, $\nabla G(u,p)$ and $-Q(t,u,p)$ all have the same direction when $p \neq 0$.

Again under the natural hypotheses (H_1)–(H_4), and also assuming (V_1)–(V_2) and (2.13), we have the following result, see [6, Theorem 2], its extension in Section 7 of [6], and the modified version of this result proved in Section 3.1 of [4].

THEOREM B. *Suppose that for all $U > 0$ there is a bounded absolutely continuous function k on J such that* (2.9), (2.12), *and*

$$(2.14) \qquad |k'| \le \begin{cases} \text{Const. } k^{2-m}, & 1 < m < 2, \\ \text{Const.}, & m \ge 2, \end{cases} \quad \textit{a.e. in } J,$$

are satisfied.

Then the rest state is a global attractor for the system (2.1).

Theorem B will be the basis for the second main theorem in Section 3. Now let

$$H(u,p) = \big(\nabla G(u,p), p\big) - G(u,p)$$

be the Legendre transform in the variable p of the action function $G(u,\cdot)$. The following observation shows that, when

$$(2.15) \qquad H(u,p) \to \infty \qquad \text{as } |p| \to \infty$$

uniformly for u in compact subsets of $\mathbb{R}^N$, then several of the earlier hypotheses can be weakened, while (2.8) is automatically satisfied.

We first recall that solutions of (2.1) have the property that

$$H(u(t), u'(t)) + F(t, u(t)) \to \text{limit} \quad \text{as } t \to \infty,$$

see [5, (3.7)] or [6, Lemma 5.1(i)]. Hence in turn, since $F(t,u) \ge 0$ by (H_2), the function

$$H(u(t), u'(t))$$

is bounded along any solution $u = u(t)$, $t \in J$. Thus by (2.15), *for any bounded solution of* (2.1) *the function $u'(t)$ is also bounded on J.*

It follows that in applying the hypotheses of Theorems A and B for any given solution of (2.1), *one can restrict consideration to compact subsets of vectors (u,p) in* $\mathbb{R}^N \times \mathbb{R}^N$. In particular, when (2.15) holds, the condition (2.5) can be weakened to the form:

> For every compact set K in $\mathbb{R}^N \times \mathbb{R}^N$ there exists a measurable control set $I \subset J$ and a number $\gamma \ge 1$ such that
>
> $$(2.5)' \qquad |Q(t,u,p)| \cdot |p| \le \gamma \left|\big(Q(t,u,p), p\big)\right| \quad \text{for all } t \in I \text{ and } (u,p) \in K.$$

Analogous restatements of (2.7) and (V_1) also hold when (2.15) is assumed. Finally (2.8) is automatically satisfied, for when (u,p) are in any given compact set the function $\nabla_u G(u,p)$ is certainly bounded.

We conclude the section with a useful estimate.

LEMMA. *Let* (2.5)–(2.7) *hold. Then for all* $U > 0$ *there is a positive constant* $c = c(U)$ *such that*

$$\delta(t) \geq c\,\sigma(t) \qquad \text{for } t \in I. \tag{2.16}$$

If (2.5)*,* (2.6) *and* (2.13) *hold, then for all* $U > 0$ *and* $\vartheta \in (0,1)$ *there is a positive constant* $d = d(U,\vartheta)$ *such that*

$$\frac{1}{|L|}\int_L \delta(t)\,dt \geq d \qquad \text{for all intervals } L \subset J \text{ with } |L| \geq \vartheta. \tag{2.17}$$

Proof. Fix $U > 0$. By (2.5) and (2.6), with $u = -p$ and $|p| = \min\{U,\ q\} = r > 0$, we get

$$\delta(t)\,|p|^{\mu} \geq -\big(Q(t,-p,p),p\big) \geq |Q(t,-p,p)| \cdot |p|/\gamma.$$

On the other hand, by (2.7)

$$|Q(t,-p,p)| \geq \sigma(t)\,|p|^{\nu},$$

proving (2.16) with $c = r^{\nu+1-\mu}/\gamma$. Next by (2.13)

$$|Q(t,-p,p)| \geq \hat{\sigma}(t)\,\varphi(-p,p),$$

so that $\delta(t) \geq \hat{d}\,\hat{\sigma}(t)$ in J, with $\hat{d} = \max\{\varphi(-p,p)\ :\ |p| = r\} \cdot r^{1-\mu}/\gamma$. In turn

$$\frac{1}{|L|}\int_L \delta(t)\,dt \geq \frac{\hat{d}}{|L|}\int_L \hat{\sigma}(t)\,dt \geq \frac{\hat{d}}{2\vartheta}\,a(\vartheta) = d > 0$$

by application of inequality (7.2) of [6] with $\lambda = \vartheta$. This completes the proof.

§3. Main Results.

Here we state our main theorems and related consequences. *It is assumed throughout, without further comment, that the conditions* (H_1)–(H_4) *are satisfied.*

Let $(I_n)_n$ be a sequence of non-overlapping intervals $I_n = [a_n, b_n]$ of J with $a_n \to \infty$, and let the control set in (H_3)–(H_4) have the form $I = \bigcup_1^\infty I_n$. We introduce the notation

$$d_n = \frac{1}{|I_n|}\int_{I_n} \delta(t)\,dt \qquad \text{(possibly } \infty\text{)},$$

which will be used throughout the paper.

Our first result is based on Theorem A of Section 2. Conditions (2.7) and (2.8) are of course required here, and also for the corresponding corollaries.

THEOREM 1. *Suppose that for every $U > 0$ there are positive constants A, B such that*

$$\sum_1^\infty \sigma_n \cdot \min\left\{|I_n|^q,\ \frac{A}{B+x_n}|I_n|\right\} = \infty, \qquad q = \begin{cases} \dfrac{m+\nu}{m-1}, & \nu \geq m-2 \\ 2, & \nu \leq m-2 \end{cases}, \tag{3.1}$$

where

$$\sigma_n = \inf_{I_n} \sigma(t) \quad \text{and} \quad x_n = \sigma_n\, d_n^\ell, \qquad \ell = 1/\mu.$$

Then $u = 0$ is a global attractor for (2.1).

The proof of Theorem 1 is given in Section 4. In the case $N = 1$, $G(p) = p^2/2$, $Q = -a(t)\,p$ and $f(u) = u$, Smith [7] obtained the weaker result that $u = 0$ is a global attractor when

$$\sum_1^\infty \sigma_n |I_n| \cdot \min\left\{|I_n|^2,\ \frac{1}{(1+\Delta_n)^2}\right\} = \infty, \qquad \Delta_n = \max_{I_n} a(t);$$

in particular for this case

$$m = 2, \quad \mu = \nu = 1, \quad q = 3, \quad \sigma(t) = a(t), \quad \delta(t) = U a(t),$$

so that taking $A = B = U$ in (3.1) we get

$$\frac{A}{B + x_n} = \frac{1}{1 + \sigma_n d_n / U} \geq \frac{1}{1 + \Delta_n^2} \geq \frac{1}{(1+\Delta_n)^2}.$$

Several special cases of Theorem 1 are of particular importance.

COROLLARY 1. *Suppose that*

$$\sup_n x_n < \infty. \tag{3.2}$$

Then $u = 0$ is a global attractor for (2.1) *if*

$$\sum_1^\infty \sigma_n \min\{|I_n|^q,\ |I_n|\} = \infty. \tag{3.3}$$

Proof. Condition (3.1) with $A = 1 + \sup_n x_n$ and $B = 1$ follows at once from (3.2), (3.3).

Clearly (3.2) holds whenever δ is bounded on $I = \bigcup_1^\infty I_n$ by (2.16). From Corollary 1 we also get the following consequence:

Suppose that

$$\sup_n x_n < \infty, \qquad \inf_n |I_n| > 0.$$

Then $u = 0$ is a global attractor if $\sum_1^\infty \sigma_n = \infty$.

A related result, applying however only for the scalar case of (1.5), appears in [3, Corollary 4.2].

COROLLARY 2. *Suppose that*

$$\inf_n x_n > 0.$$

Then $u = 0$ *is a global attractor for* (2.1) *if*

$$\sum_1^\infty d_n^{-\ell} \ \min\{|I_n|^q, |I_n|\} = \infty, \quad \textit{where } \ell = 1/\mu. \tag{3.4}$$

Proof. Condition (3.1) with $A = 1 + \inf_n x_n$, $B = 1$ follows easily from (3.4) together with the relations

$$\sigma_n = x_n d_n^{-\ell} \geq x\, d_n^{-\ell}, \qquad \frac{\sigma_n A}{1 + x_n} = \frac{x_n(1+x)}{1+x_n}\, d_n^{-\ell} \geq x\, d_n^{-\ell},$$

where $x = \inf x_n > 0$.

COROLLARY 3. *Suppose that*

$$\inf_I \sigma(t) > 0, \qquad \sup_n d_n < \infty. \tag{3.5}$$

Then $u = 0$ *is a global attractor for* (2.1) *if*

$$\sum_1^\infty |I_n|^q = \infty. \tag{3.6}$$

Proof. This is an immediate consequence of Corollary 1 or Corollary 2. For example, (3.5) implies that

$$\inf_n x_n \geq c^\ell (\inf_n \sigma_n)^{1+\ell} > 0 \quad \text{and} \quad \inf_n d_n^{-\ell} \geq (\sup_n d_n)^{-\ell} > 0,$$

where c is the constant in (2.16). Hence by Corollary 2 the rest state is a global attractor provided that

$$\sum_1^\infty \min\{|I_n|^q, \, |I_n|\} = \infty.$$

But this series diverges if and only if (3.6) diverges.

Our second main result is based on Theorem B. In this case conditions (2.13) and (V_1)–(V_2) are required (instead of (2.7)–(2.8)). We recall again that the control set has the form $I = \overset{\infty}{\underset{1}{\cup}} I_n$.

THEOREM 2. *Suppose that for every $U > 0$ there exists a positive constant A such that*

$$(3.7)\qquad \sum_1^\infty \min\left\{|I_n|^{\bar q},\quad \frac{A}{d_n^\ell}\,|I_n|\right\} = \infty, \qquad \bar q = \begin{cases} \dfrac{m}{m-1}, & 1 < m \le 2 \\ 2, & m \ge 2 \end{cases}.$$

Then $u = 0$ is a global attractor for (2.1).

The proof of Theorem 2 is given in Section 5. The hypotheses of Theorem 2 hold for the system (1.5) when A has the decomposition (1.7), see the comments at the end of the section.

Theorem 2 has the following consequences.

COROLLARY 4. *Suppose that*

$$\sup_n d_n < \infty.$$

Then $u = 0$ is a global attractor for (2.1) *if*

$$(3.8)\qquad \sum_1^\infty |I_n|^{\bar q} = \infty.$$

Proof. Taking $A = (\sup_n\ d_n)^\ell$, we see that the series in (3.7) is greater than

$$\sum_1^\infty \min\left\{|I_n|^{\bar q},\ |I_n|\right\}.$$

This diverges if and only if $\sum_1^\infty |I_n|^{\bar q}$ diverges (since $\bar q > 1$).

For the canonical case $m = 2$, $\mu = 1$, $\nu = 1$, the exponents q and $\bar q$ in Corollaries 3 and 4 have the respective values 3 and 2, these being best possible as shown in Section 6.

COROLLARY 5. *Suppose there is a positive constant such that*

$$(3.9)\qquad d_n \ge \text{Const.} \begin{cases} |I_n|^{-\mu} & \textit{if} \quad m \ge 2 \\ |I_n|^{-\mu/(m-1)} & \textit{if}\ \ 1 < m \le 2. \end{cases}$$

Then $u = 0$ is a global attractor for (2.1) *if*

$$(3.10)\qquad \sum_1^\infty |I_n|\, d_n^{-\ell} = \infty.$$

Proof. Let the constant in (3.9) be denoted by D, and choose $A = D^\ell$ in (3.7). Then the second term in braces in (3.7) is less than the first, so that (3.10) implies (3.7).

COROLLARY 6. (Criterion of Thurston–Wong type). *Let* $\inf_n |I_n| > 0$. *Then* $u = 0$ *is a global attractor for* (2.1) *if* (3.10) *is satisfied.*

Proof. From (2.17) we have $d_n \ge d > 0$ for all n, and in turn (3.9) obviously holds because $\inf_n |I_n| > 0$.

Thurston & Wong discovered the special case of Corollary 6 when $N = 1$, $|I_n| = 1$, $G(p) = p^2/2$, $Q(t,u,p) = -a(t,u,p)\,p$, and f is independent of t. Their assumptions imply $\mu = \ell = 1$, in which case (3.10) takes exactly their form $\sum_1^\infty \left(\int_{I_n} \delta(t)\,dt\right)^{-1} = \infty$.

Artstein & Infante [1, condition (2.7)] showed for the same case that $u = 0$ is a global attractor provided

$$\frac{1}{K^2}\sum_1^K d_n \le B$$

for some constant B independent of K. In fact, more generally, without any restrictions on the measures of I_n, and whatever the value of μ, condition (3.10) is implied by

$$\text{(3.11)} \qquad \frac{1}{K^{\mu+1}}\sum_1^K c_n \le B, \qquad \text{where} \quad c_n = \frac{1}{|I_n|^{\mu+1}}\int_{I_n}\delta(t)\,dt = \frac{d_n}{|I_n|^{\mu}}.$$

To see this, note that for any positive integers $0 < L < K$, we have by Hőlder's inequality

$$K - L = \sum_L^K 1 \le \left(\sum_L^K c_n\right)^{1/(\mu+1)} \left(\sum_L^K c_n^{-1/\mu}\right)^{\mu/(\mu+1)},$$

so that in turn

$$\text{(3.12)} \qquad \sum_L^K c_n^{-\ell} \ge \left[\frac{1}{(K-L)^{\mu+1}}\sum_L^K c_n\right]^{-1/\mu}.$$

But, by (3.11), if $K = 2L$ then

$$\frac{1}{(K-L)^{\mu+1}}\sum_L^K c_n \le 2^{\mu+1}B.$$

Hence from (3.12) it follows that

$$\sum_1^{2^\nu} c_n^{-\ell} \ge \frac{\nu}{2(2B)^\ell}, \qquad \nu = 1, 2, \dots .$$

Thus the series (3.10) diverges. We have proved the following

COROLLARY 7. (Criterion of Artstein–Infante type). *Suppose* $\inf_n |I_n| > 0$*, or more generally that* (3.9) *holds. Then* $u = 0$ *is a global attractor for* (2.1) *if* (3.11) *is satisfied.*

In essentially the same way, condition (3.4) in Corollary 2 can also be deduced from the Artstein–Infante type condition (3.11), provided $\inf_n |I_n| > 0$.

Remark. When $\inf_n |I_n| > 0$ the criteria (3.4) of Corollary 2 and (3.10) of Corollary 6 are equivalent. Since by (2.16) the condition $\inf_n x_n > 0$ holds whenever $\inf_I \sigma(t) > 0$, one can see a connection between the hypotheses of these corollaries. On the other hand, the assumptions of Theorem 2 are enough different from those of Theorem 1 that the corollaries are not directly comparable.

The system (1.5)**.**

We show that Corollaries 3 and 4 apply to the system (1.5). For Corollary 3 the hypotheses of Theorem A must be verified, on the basis of the assumptions immediately following (1.5).

Fix a compact subset K of $\mathbb{R}^N \times \mathbb{R}^N$, and let $\alpha > 0$ be such that

$$\big(A(t,u,p)p,p\big) \geq \alpha\,|p|^2 \qquad \text{for } (t,u,p) \in I \times K;$$

also denote by $\|A\|$ the L^∞ norm of A on $I \times K$. Then one easily sees that (2.5) holds in $I \times K$ with $\gamma = \alpha/\|A\|$, that (2.6) is satisfied with $\delta(t) = U\,\|A\| = \text{Const.}$ and $\mu = 1$, and that (2.7) is verified with $\sigma(t) = \alpha$, $\nu = 1$. Finally, taking into account the observation at the end of Section 2, we see that Theorem A is applicable to (1.5), with $m = 2$.

In turn, since $\nu = 1$, we get $q = (m+\nu)/(m-1) = 3$ in (3.1). Moreover $\inf_I \sigma(t) = \alpha > 0$ and $\sup_n d_n = U\,\|A\| < \infty$, so (3.5) is satisfied. Corollary 3 then gives the criterion (1.6).

For Corollary 4 the hypotheses of Theorem B must be verified on the basis of the assumptions (1.7)–(1.9). Again fix a compact set K of $\mathbb{R}^N \times \mathbb{R}^N$. Then one easily sees that

$$\big(A(t,u,p)p,p\big) = \beta(t,u,p)\,|p|^2 + \big(B(t,u,p)p,p\big) \geq \beta_1\,|p|^2 \qquad \text{on } J \times K,$$

by (1.8) and the fact that B is non–negative definite. Hence we can take $\hat{\sigma}(t) = 1$ and $\varphi(u,p) = \beta_1\,|p|^2$ in (2.13). Also $\|A\| \leq \beta_2 + \|B\| < \infty$ by (1.9). Thus (2.5) holds in $I \times K$ with $\gamma = \beta_1/\|A\|$. As before we can take $\delta(t) = U\,\|A\|$ and $\mu = 1$.

Next (V_1) is satisfied with $h(t) = \beta_1 p_0^2$. Finally, for (V_2),

$$\big(Q(t,u,p),u\big) = -\beta(t,u,p)\,(p,u) - \big(B(t,u,p)p,u\big) \leq U\,\|B\|\,|p|,$$

when $|u| \leq U$ and $(p,u) \geq 0$. This gives the first part of (V_2) with $\varepsilon(p) = U\,\|B\|\,|p|$. The second part of (V_2) is automatic for (1.5) with $g(u) = \frac{1}{2}u^2$ and $C = 0$. Again taking into account the observation at the end of Section 2, we see that Theorem B is applicable.

As before $\sup_n d_n = U\,\|A\| < \infty$. Corollary 4 then gives the criterion (1.10), since $m = 2$ and $\bar{q} = 2$ by (3.7).

§4. Proof of Theorem 1.

Recall that $I = \bigcup_{1}^{\infty} I_n$ where $I_n = [a_n,\ b_n]$. We begin with a simple

LEMMA. *Let k be non–negative measurable function such that $k(t) = 0$ for $t \in J \setminus I$ and for which*

$$d_n\, k_n^{\mu} \le M_1 \tag{4.1}$$

and

$$\int_{I_n} k(t)dt \ge \frac{1}{M_2}\, |I_n|\, k_n, \tag{4.2}$$

where $k_n = \sup_{I_n} k(t)$ and $M_1,\ M_2$ are positive constants.

Then k satisfies condition (2.12) *with $M = M_1 M_2$.*

Proof. We have

$$\begin{aligned} \int_{I_n} \delta(t) k^{\mu+1}(t)dt &\le k_n^{\mu+1} \int_{I_n} \delta(t)dt = k_n |I_n|\, k_n^{\mu} d_n \\ &\le k_n |I_n| M_1 && \text{by } (4.1) \\ &\le M_1 M_2 \int_{I_n} k(t)dt && \text{by } (4.2). \end{aligned}$$

Condition (2.12) now follows by summation over n.

Proof of Theorem 1. Recall that $x_n = \sigma_n\, d_n^{\ell}$, $\ell = 1/\mu$, $q = (\nu + m)/(m-1)$ if $\nu \ge m-2$ and $q = 2$ if $\nu \le m-2$.

We now construct a bounded piecewise smooth function $k = k(t)$ satisfying the assumptions $(2.9)_2$ and (2.10) of Theorem A. In particular, let $k = 0$ on $J \setminus I$. To obtain k on the intervals I_n, we consider separately the two subcases

$$|I_n|^{q-1} \le \frac{A}{B + x_n} \tag{i}$$

and

$$|I_n|^{q-1} > \frac{A}{B + x_n}. \tag{ii}$$

Subcase (i). Let $I_n = [a_n, b_n]$ and put

$$k(t) = \begin{cases} C\, \sigma_n (t - a_n)^{q-1}, & a_n \le t \le \frac{1}{2}(a_n + b_n) \\ \\ C\, \sigma_n (b_n - t)^{q-1}, & \frac{1}{2}(a_n + b_n) \le t \le b_n, \end{cases}$$

where $C = 2^{q-1}B/A$. Then $(2.10)_2$ holds on I_n with the Const. $= 2\,(q-1)\,(B/A)^{\lambda}$, independent of n. The exponent $\lambda = 1/(q-1)$ satisfies (2.11) since q is given by $(3.1)_2$. Next, letting $k_n = \max\limits_{I_n} k(t)$ as in the lemma, we have

$$k_n = \; k\left(\frac{a_n+b_n}{2}\right) = C\,\sigma_n \left(\frac{|I_n|}{2}\right)^{q-1} \le 2^{1-q} C\,\sigma_n \,\frac{A}{B+x_n} = \frac{B\sigma_n}{B+x_n} \tag{4.3}$$

by (i) and the choice of C. In turn, since $x_n = \sigma_n\, d_n^{\ell} \ge c^{\ell}\, \sigma_n^{\ell+1}$ by (2.16), there follows

$$k_n \le \frac{B\sigma_n}{B + c^{\ell}{\sigma_n}^{\ell+1}} \le D, \tag{4.4}$$

where D is a constant depending only on B, c and ℓ. Hence k is uniformly bounded on intervals I_n of type (i), with the bound independent of n. Moreover, again from (4.3), we have $k(t) \le \sigma_n \le \sigma(t)$, so $(2.10)_1$ holds on these intervals with Const. $= 1$.

Subcase (ii). Put

$$k(t) = \begin{cases} C\,\sigma_n (t-a_n)^{q-1}, & a_n \le t \le t_n \\[2ex] \dfrac{B\sigma_n}{B+x_n}, & t_n < t < \bar{t}_n \\[2ex] C\,\sigma_n (b_n - t)^{q-1}, & \bar{t}_n \le t \le b_n, \end{cases}$$

where t_n and $\bar{t}_n$ are chosen so that k is continuous on I_n. This can be done because of condition (ii).

As before k satisfies $(2.10)_2$ on I_n with the same Const. $= 2\,(q-1)\,(B/A)^{\lambda}$, since $k' = 0$ on $(t_n, \bar{t}_n)$. Moreover

$$k_n = \frac{B\sigma_n}{B+x_n} \le D, \tag{4.5}$$

as in (4.4). Therefore $(2.10)_1$ is satisfied and k is uniformly bounded on intervals I_n of type (ii).

We next show that k satisfies conditions (4.1) and (4.2) of the lemma. Indeed by (4.3) and (4.5), for each n,

$$d_n k_n^{\mu} \le d_n \left(\frac{B\,\sigma_n}{B+x_n}\right)^{\mu} = B^{\mu}\left(\frac{x_n}{B+x_n}\right)^{\mu} \le B^{\mu} = M_1.$$

Thus (4.1) is verified.

In case (i) an easy calculation and the use of (4.3) gives

$$\int_{I_n} k(t)\ dt = \frac{2}{q}\, C\, \sigma_n \left(\frac{|I_n|}{2}\right)^q = \frac{1}{q}\, |I_n|\, k_n,$$

while in case (ii)

$$\begin{aligned}\int_{I_n} k(t)\ dt &= \frac{(t_n - a_n)}{q}\, k(t_n) + (\bar{t}_n - t_n)\frac{B\,\sigma_n}{B + x_n} + \frac{(b_n - \bar{t}_n)}{q}\, k(\bar{t}_n) \\ &= \left\{\frac{1}{q}(t_n - a_n) + (\bar{t}_n - t_n) + \frac{1}{q}(b_n - \bar{t}_n)\right\}\frac{B\,\sigma_n}{B + x_n} \qquad (4.6) \\ &> \frac{1}{q}\, |I_n|\, k_n,\end{aligned}$$

since $q > 1$. Hence (4.2) holds with $M_2 = q$.

The lemma now shows that condition (2.12) of Theorem A is satisfied. It remains only to verify the hypothesis $(2.9)_1$ to finish the first part of the proof. We already know that in case (i)

$$\int_{I_n} k(t)\ dt = \frac{2}{q}\, C\, \sigma_n \left(\frac{|I_n|}{2}\right)^q = \frac{1}{q}\,\frac{B}{A}\,\sigma_n\, |I_n|^q$$

by the choice of C, while in case (ii) by (4.5) and (4.6)

$$\int_{I_n} k(t)\ dt > \frac{1}{q}\,\sigma_n\, |I_n|\, \frac{B}{B + x_n}.$$

Consequently for each n

$$\int_{I_n} k(t)\ dt \geq \frac{1}{q}\,\frac{B}{A}\,\sigma_n\, \min\left\{|I_n|^q,\ \frac{A}{B + x_n}\, |I_n|\right\}.$$

Since

$$\int_J k(t)\ dt = \sum_1^\infty \int_{I_n} k(t)\ dt,$$

condition (3.1) now shows that $k \notin L^1(J)$, as required in $(2.9)_1$.

This completes the proof.

§5. Proof of Theorem 2.

From (2.17) we have $d_n \geq d > 0$ for all n.

As in the proof of Theorem 1, we shall construct a bounded piecewise smooth function $k = k(t)$ satisfying the assumptions $(2.9)_2$ and (2.14) of Theorem B. In particular, we define k to be 0 on $J \setminus \bigcup_1^\infty I_n$ and, to obtain k on the intervals I_n, consider separately the two cases:

$$\text{(i)} \qquad |I_n|^{\bar{q}-1} \leq \frac{A}{d_n^\ell}$$

and

$$\text{(ii)} \qquad |I_n|^{\bar{q}-1} > \frac{A}{d_n^\ell}.$$

Case (i). Put

$$k(t) = \begin{cases} C(t-a_n)^{\bar{q}-1}, & a_n \leq t \leq \frac{1}{2}(a_n+b_n) \\ \\ C(b_n-t)^{\bar{q}-1}, & \frac{1}{2}(a_n+b_n) \leq t \leq b_n, \end{cases}$$

where $C = 2^{\bar{q}-1}/A$. Then recalling the definition of $\bar{q}$ in (3.7), we see that (2.14) holds on I_n with Const. $= (\bar{q}-1)C^{m-1}$ when $1 < m < 2$ and Const. $= C$ when $m \geq 2$.

Next

$$k_n = k\left(\tfrac{1}{2}(a_n+b_n)\right) = C\left(\tfrac{1}{2}|I_n|\right)^{\bar{q}-1} \leq d_n^{-\ell} \leq d^{-\ell} \tag{5.1}$$

by (i) and the choice of C. Hence k is bounded on each I_n of type (i), uniformly in n.

Case (ii). Put

$$k(t) = \begin{cases} C(t-a_n)^{\bar{q}-1}, & a_n \leq t \leq t_n \\ d_n^{-\ell}, & t_n < t < \bar{t}_n \\ C(b_n-t)^{\bar{q}-1}, & \bar{t}_n \leq t \leq b_n, \end{cases}$$

where t_n and $\bar{t}_n$ are chosen so that k is continuous on I_n. This can be done in virtue of (ii). As in case (i), we see that (2.14) is satisfied and that k is bounded on I_n uniformly in n, namely $k_n = d_n^{-\ell} \leq d^{-\ell}$.

We next show that k satisfies conditions (4.1) and (4.2) of the lemma in Section 4. Indeed for each n

$$d_n\, k_n^\mu \leq d_n (d_n^{-\ell})^\mu = 1 = M_1.$$

Thus (4.1) is verified.

In case (i) by (5.1)

$$\int_{I_n} k(t)\ dt = 2\frac{C}{\bar{q}}\left(\frac{1}{2}|I_n|\right)^{\bar{q}} = \frac{1}{\bar{q}}\ |I_n|\ k_n,$$

while in case (ii), as in the proof of Theorem 1,

$$\int_{I_n} k(t)\ dt > \frac{1}{\bar{q}}\ |I_n|\ k_n.$$

Hence (4.2) holds with $M_2 = \bar{q}$.

The lemma now shows that condition (2.12) is satisfied, so that to apply Theorem B it remains only to verify $(2.9)_1$. Arguing as in the proof of Theorem 1, we obtain

$$\int_{I_n} k(t)\ dt \geq \frac{1}{\bar{q}A}\ \min\left\{|I_n|^{\bar{q}},\ \frac{A}{d_n^{\ell}}|I_n|\right\}.$$

Consequently (3.7) implies that $k \notin L^1(J)$ and this completes the proof.

§6. Examples.

The purpose of this section is to show that the exponents which appear in conditions (1.6) and (1.10) are best possible.

Consider the linear equation

$$u'' + a(t)u' + u = 0, \qquad t \in J = [0, \infty), \tag{6.1}$$

where $a\colon J \to \mathbb{R}$ is an *on–off* damping function of the form

$$a(t) = \begin{cases} 0 & \text{in } J \setminus \bigcup_1^\infty I_n \\ 2 & \text{in } \bigcup_1^\infty I_n. \end{cases} \tag{6.2}$$

The following result shows that the exponent 3 in (1.6) is best possible.

PROPOSITION 1. *Let $\epsilon \in (0, 2]$ be fixed and let $(\lambda_n)_n$ be a sequence of positive real numbers such that*

$$\sum_1^\infty \lambda_n^{3-\epsilon} = \infty, \qquad \sum_1^\infty \lambda_n^3 < \infty. \tag{6.3}$$

Then there exists a sequence of disjoint intervals $I_n = [a_n,\ a_n + \lambda_n]$ *such that* $u = 0$ *is not a global attractor for* (6.1) *with the damping* (6.2).

Remarks. Since the damping function (6.2) is not continuous the corresponding solutions of (6.1) must be sought in the class $C^1(J)$. Of course, a smoothing procedure will obviously yield a corresponding result for (6.1) with continuous damping.

From the proof it will be clear that the sequence $(I_n)_n$ can be chosen to have arbitrarily large gaps, i.e., with $a_{n+1} - a_n$ unbounded.

Proof. We place the interval I_1 arbitrarily in $J = [0, \infty)$, and recursively determine the location of the successive intervals I_n for $n \geq 2$. In particular we shall impose the Cauchy conditions

$$u(a_n) = A_n, \qquad u'(a_n) = 0, \qquad A_1 \neq 0 \tag{6.4}$$

in order to construct a bounded solution u of (6.1)–(6.2) which does not approach 0 as $t \to \infty$. The values A_n will be recursively determined, along with the location of the intervals I_n.

From (6.4) and (6.2) it is clear that

$$u(t) = A_n(1 + t - a_n)e^{a_n - t} \qquad \text{for} \quad t \in I_n. \tag{6.5}$$

On the other hand, in the intervals $(a_n + \lambda_n,\ a_{n+1})$ between the sets I_n and I_{n+1}, we have

$$u(t) = \varphi_n(t) = B_n \cos(t + \theta_n) \tag{6.6}$$

for some constants B_n and θ_n, again by (6.2). Clearly (6.6) should join smoothly with (6.5) at the point $a_n + \lambda_n$ and also satisfy the conditions $\varphi_n(a_{n+1}) = A_{n+1}$ and $\varphi_n'(a_{n+1}) = 0$. These latter conditions take the specific form

$$B_n \cos(a_{n+1} + \theta_n) = A_{n+1}, \qquad B_n \sin(a_{n+1} + \theta_n) = 0,$$

so that $B_n^2 = A_{n+1}^2$. The former conditions are

$$\begin{aligned} B_n \cos(a_n + \lambda_n + \theta_n) &= A_n(1 + \lambda_n)e^{-\lambda_n} \\ B_n \sin(a_n + \lambda_n + \theta_n) &= A_n \lambda_n e^{-\lambda_n}. \end{aligned}$$

Squaring and adding gives

$$A_{n+1}^2 = A_n^2\{(1 + \lambda_n)^2 + \lambda_n^2\}e^{-2\lambda_n} \equiv A_n^2 \Phi(\lambda_n). \tag{6.7}$$

This determines A_{n+1}^2 in terms of A_n^2 and λ_n. Because θ_n is not yet chosen, it is clear that B_n and A_{n+1} are so far determined only up to their signs. We can choose the sign of B_n as we wish, say sign $B_n =$ sign A_{n+1}. Then $\cos(a_{n+1} + \theta_n) = 1$, so without loss of generality $\theta_n = -a_{n+1}$. In turn

$$\tan(a_{n+1} - a_n - \lambda_n) = -\frac{\lambda_n}{1 + \lambda_n},$$

which determines a_{n+1} modulo π; indeed if sign $A_{n+1} =$ sign A_n then one sees that

$$a_{n+1} - a_n - \lambda_n \in \left(\tfrac{7}{4}\pi,\ 2\pi\right) \qquad \text{modulo } 2\pi,$$

while if sign $A_{n+1} = -$sign A_n then

$$a_{n+1} - a_n - \lambda_n \in \left(\tfrac{3}{4}\pi,\ \pi\right) \qquad \text{modulo } 2\pi.$$

Clearly $a_{n+1} - a_n$ can be arbitrarily large, though not arbitrarily small; in fact since $\lambda_n \to 0$, there is a sequence $(k_n)_n$ of positive integers such that $\lim_n (a_{n+1} - a_n - k_n\pi) = 0$.

An easy calculation shows that

$$1 - \tfrac{4}{3}x^3 < \Phi(x) < 1 \qquad \text{for} \quad x > 0.$$

Hence $|A_{n+1}| < |A_n|$ by (6.7), so that

$$\limsup_{t\to\infty} |u(t)|^2 = \lim_n A_n^2 = A_1^2 \prod_1^\infty \Phi(\lambda_n).$$

We can assume without loss of generality that $\lambda_n^3 < 3/4$ for all n, in virtue of $(6.3)_2$. Therefore,

$$\limsup_{t\to\infty} |u(t)|^2 > A_1^2 \prod_1^\infty \left(1 - \tfrac{4}{3}\lambda_n^3\right) > 0,$$

where the last inequality is equivalent to $\sum_1^\infty \lambda_n^3 < \infty$. This completes the demonstration.

The proof sharply brings out the role of the exponent 3 in condition (1.6). Figures 1 shows a typical graph of u with all the $A_n > 0$ and with varying spacing between the I_n.

FIGURE 1. THE SOLUTION u OF PROPOSITION 1.

In Figure 2 the heavy curve is the solution (6.5). The light curve is one arch of the cosine wave φ_n defined by (6.6), whose amplitude is A_{n+1}. The dashed curve is one arch of the cosine wave φ_{n-1}, whose amplitude is A_n.

FIGURE 2. BEHAVIOR OF u NEAR AN INTERVAL I_n.

The next result shows that (1.6) is *not* necessary for the global stability of the rest state of (6.1) when

$$\inf_n \sigma_n > 0 \quad \text{and} \quad \sup_n d_n < \infty.$$

It also indicates the extreme delicacy of the situation when one has *on–off* damping, that is, the exact switching times can be of great importance.

PROPOSITION 2. *Under the hypotheses of Proposition* 1 *there also exists a sequence of disjoint intervals* $I_n = [a_n, a_n + \lambda_n]$, *with* $a_{n+1} - a_n > \pi$, *such that* $u = 0$ *is a global attractor for* (6.1) *with the damping* (6.2).

Proof. We place I_1 arbitrarily. To construct the remaining intervals, choose some sequence $(k_n)_n$ of positive integers and define

$$a_{n+1} = a_n + \lambda_n + k_n\pi.$$

Now let u be any solution of (6.1)–(6.2) on J. Since (6.6) holds on the intervals $(a_n + \lambda_n, a_{n+1})$, whose lengths are multiples of π, it is evident that

$$\begin{aligned} u(a_{n+1}) &= (-1)^{k_n}\; u(a_n + \lambda_n) \\ u'(a_{n+1}) &= (-1)^{k_n}\; u'(a_n + \lambda_n). \end{aligned} \tag{6.8}$$

Now, for $\tau \in [0, \infty)$, we put

$$v(\tau) = (-1)^{j_n}\, u(a_n + \tau - \ell_n) \quad \text{if } \tau \in [\ell_n, \ell_{n+1}), \quad n = 1, 2, \ldots,$$

where $j_n = \sum_1^{n-1} k_i$ and $\ell_n = \sum_1^{n-1} \lambda_i$ when $n \geq 2$, and $j_1 = \ell_1 = 0$. By $(6.3)_1$ it is clear that $\ell_n \to \infty$ as $n \to \infty$, so v is well defined. Also v is of class C^1 in view of (6.8).

Directly from the definition of v we see that v is a solution of

$$v'' + 2v' + v = 0 \qquad \text{in} \ \ [0, \infty), \tag{6.9}$$

so that v is certainly of class C^2. The equation (6.9) has characteristic values $r_1 = r_2 = -1$, hence $v(\tau) \to 0$ and $v'(\tau) \to 0$ as $\tau \to \infty$.

From this we get

$$\sup_{I_n} |u(t)| \ \to \ 0, \qquad \sup_{I_n} |u'(t)| \ \to \ 0, \qquad \text{as } n \to \infty;$$

it then follows at once that the coefficients B_n in (6.6) also approach 0 as $n \to \infty$. This completes the proof.

Remark. Similar conclusions can be obtained for the equation

$$u'' + a(t)\,|u'|^{\nu-1}u' + u = 0, \qquad \nu \neq 1,$$

but we shall not pursue this here.

To investigate the exponent 2 in condition (1.10), we consider equation (6.1) with the singular *on–off* damping

$$a(t) = \begin{cases} \infty & \text{in } J \setminus \bigcup_1^\infty I_n \\ 2 & \text{in } \bigcup_1^\infty I_n. \end{cases} \tag{6.10}$$

Of course a is neither continuous nor has its values in the reals, and equation (6.1) on $J \setminus \bigcup_1^\infty I_n$ must therefore be interpreted in terms of a family of equations in which the damping uniformly approaches ∞ on any compact (or even any bounded) subset of $J \setminus \bigcup_1^\infty I_n$. The corresponding solutions approach constants on the open intervals between the I_n's. In fact, the appropriate interpretation of a solution of the initial value problem $u(a_n + \lambda_n) = \alpha$, $u'(a_n + \lambda_n) = \beta$ on the interval $J_n = (a_n + \lambda_n,\ a_{n+1}]$ is $u(t) \equiv \alpha$. That is, if u_M is the solution of the initial value problem

$$\begin{cases} u'' + Mu' + u = 0 \\ u_M(a_n + \lambda_n) = \alpha, \qquad u'_M(a_n + \lambda_n) = \beta \end{cases}$$

on the interval $\overline{J_n}$, then $u_M(t) \to \alpha$ as $M \to \infty$ uniformly on $\overline{J_n}$, while $u'_M(t) \to 0$ uniformly on any compact subset of J_n. Accordingly, solutions of (6.1), (6.10) are to be interpreted as functions of class

$$C(J) \cap C^1(J \setminus \{a_n + \lambda_n,\ n = 1, 2, \dots\})$$

which satisfy (6.1) with $a(t) = 2$ on each interval $[a_n,\ a_n + \lambda_n)$ and which are constant on each interval $\overline{J_n}$.

PROPOSITION 3. *Let* $(I_n)_n$ *be a sequence of disjoint intervals* $I_n = [a_n,\ a_n + \lambda_n]$. *Then* $u = 0$ *is a global attractor for* (6.1) *with the damping* (6.10) *if and only if*

$$\sum_1^\infty |I_n|^2 = \infty.$$

Proof. If $\sum\limits_1^\infty |I_n|^2 = \infty$ then by Corollary 4 the rest state $u = 0$ is a global attractor.

Now assume that $\sum\limits_1^\infty |I_n|^2 < \infty$ and, without loss of generality, that $\lambda_n^2 < 2$. For simplicity we first consider the case

$$a_n \nearrow \infty \qquad \text{as } n \to \infty. \tag{6.11}$$

Then every solution of (6.1), (6.10) on J has the form

$$u(t) = \begin{cases} A_1 & \text{if } t \in [0, a_1] \\ A_n(1 + t - a_n)e^{a_n - t} & \text{if } t \in I_n \\ A_{n+1} & \text{if } t \in J_n. \end{cases}$$

Furthermore, by continuity at the point $a_n + \lambda_n$ we have the recursive formula

$$A_{n+1} = A_n(1 + \lambda_n)e^{-\lambda_n}.$$

Obviously

$$1 - \tfrac{1}{2}x^2 < (1 + x)e^{-x} < 1 \qquad \text{for} \quad x > 0.$$

It follows that $(|A_n|)_n$ is decreasing and so also $|u(t)|$ is decreasing on $[0, \infty)$. Thus

$$\lim_{t\to\infty} |u(t)| = |A_1| \cdot \prod_1^\infty (1 + \lambda_n)e^{-\lambda_n}.$$

Then if $A_1 \neq 0$

$$\lim_{t\to\infty} |u(t)| > |A_1| \cdot \prod_1^\infty (1 - \tfrac{1}{2}\lambda_n^2) > 0.$$

Consequently every solution, except the trivial one ($A_1 = 0$), approaches a non–zero limit at ∞. In particular, $u = 0$ is not a global attractor for (6.1), (6.10).

If (6.11) fails, then $a_n \nearrow$ finite a. The previous proof shows that $u(t) \to u_0 \neq 0$ as $t \nearrow a$, so that in turn

$$u(t) \equiv u_0 \qquad \text{for } t \geq a;$$

the solution of course need not be smooth at the point $t = a$. This completes the proof.

Remark. Proposition 1 shows that the condition $\delta \notin L^1(J)$ alone is *not* sufficient for the rest state to be a global attractor, though it *is* known to be a necessary condition, see [5, Section 5, Corollary 1]. Indeed for (6.1)–(6.2) the hypotheses of this corollary are satisfied, with $\hat{\delta}(t) = 2\,a(t)$; see [5, condition (5.3)]. Moreover in this example $\delta(t) = U\,a(t) = \frac{1}{2}\,U\,\hat{\delta}(t)$. But by (6.2)

$$\int_J a(t)\;dt = \sum_1^\infty \int_{I_n} 2\;dt = 2\sum_1^\infty\;|I_n|,$$

and the last series diverges by $(6.3)_1$ – obviously if $|I_n| \geq 1$ for an infinite number of integers n, and otherwise since

$$\sum_K^\infty |I_n| \;\geq\; \sum_K^\infty |I_n|^{3-\epsilon} = \infty.$$

for K suitably large. Thus $\delta \notin L^1(J)$, but nevertheless by Proposition 1 the rest state is not a global attractor.

Acknowledgment. The first author is a member of *Gruppo Nazionale di Analisi Funzionale e sue Applicazioni* of the *Consiglio Nazionale delle Ricerche.* This research has been partly supported by the Italian *Ministero della Università e della Ricerca Scientifica e Tecnologica.*

REFERENCES

[1] Z. ARTSTEIN & E.F. INFANTE, *On the asymptotic stability of oscillators with unbounded damping*, Quart. Applied Math. **35** (1976), 195–199.

[2] L. HATVANI, *Nonlinear oscillation with large damping*, to appear in Dynamical Systems and Applications.

[3] L. HATVANI & V. TOTIK, *Asymptotic stability of the equilibrium of the damped oscillator*, to appear in Diff. Integral Equations.

[4] G. LEONI, *On a theorem of Pucci and Serrin*, to appear in J. Diff. Equations.

[5] P. PUCCI & J. SERRIN, *Precise damping conditions for global asymptotic stability for nonlinear second order systems*, to appear in Acta Math..

[6] P. PUCCI & J. SERRIN, *Precise damping conditions for global asymptotic stability for nonlinear second order systems, II*, to appear in J. Diff. Equations.

[7] R.A. SMITH, *Asymptotic stability of* $x'' + a(t)x' + x = 0$, Quart. J. Math. Oxford **12** (1961), 123–126.

[8] L.H. THURSTON & J.W. WONG, *On global asymptotic stability of certain second order differential equations with integrable forcing terms*, SIAM J. Appl. Math. **24** (1973), 50– 61.

Patrizia Pucci
Dipartimento di Matematica
Università di Perugia
Italy

James Serrin
Department of Mathematics
University of Minnesota
Minneapolis