Some Remarks on Global Nonexistence for Nonautonomous Abstract Evolution Equations

Howard A. Levine   Patrizia Pucci   James Serrin

Abstract. In this paper we consider the problem of non-continuation of solutions of the initial value problem for abstract evolution equations of the form

$$Pu_{tt} + Q(t)u_t + A(t,u) = F(t,u), \quad t \in J = [0,\infty)$$

where P and Q are linear self-adjoint operators, and $A(t,u)$ and $F(t,u)$ are respectively a linear operator in u (typically of differential type) and a nonlinear driving force. We also consider the case in which the linear dissipation, $Q(t)u_t$ is replaced by a nonlinear dissipation, $Q(t,u,u_t)$. The results extend earlier works of Levine and Levine and Serrin.

§1. Introduction.

In a series of papers [1–3] by the first author the problem of non–continuation was studied for abstract evolution equations of the type

$$Pu_{tt} + Qu_t + A(t,u) = F(u), \quad t \in J = [0,\infty), \tag{1.1}$$

where P and Q are linear self-adjoint operators, and $A(t,u)$ and $F(u)$ are respectively a linear operator in u (typically of differential type) and a nonlinear driving force. In [4] the operator A was allowed some degree of nonlinearity, but at the expense of dropping dependence on t.

In a recent paper [5] this work was extended to evolution equations of the form

$$(P(u_t))_t + Q(t,u,u_t) + A(u) = F(u), \tag{1.2}$$

where all the operators are allowed to be nonlinear, though with A independent of time.

The purpose of the present paper is to generalize the above results, and to some extent to unify them, by admitting *the same degree of nonlinearity* for the operators A and F in both (1.1) and (1.2), by letting both A *and* F depend on time, and finally by allowing time–dependent operators Q in (1.1).

Our results are established in two theorems, the first (Section 2) corresponding to the ideas and methods of [1–3], and the second (Section 3) to those of [5]. We end with the presentation of various examples, of interest in themselves, illustrating the range of our conclusions when interpreted in concrete cases. In this respect it almost goes without saying that the results of both Sections 2 and 3 apply to the wave operator $u_{tt} - \Delta u$, or its degenerate counterpart $u_{tt} - \mathrm{div}(|Du|^{s-2}Du)$, and to the corresponding parabolic operators $u_t - \Delta u$ and $u_t - \mathrm{div}(|Du|^{s-2}Du)$.

While the results of Section 2 are, in some ways, more special than those of Section 3, we note in Section 4 several situations of interest covered by Theorem 1 but not by Theorem 2.

§2. The case with linear dissipation.

Let X be a Banach space, and X' its dual space. If $x \in X$ and $x' \in X'$, we shall write $\langle x', x\rangle_X$ to denote the natural pairing of x and x', that is $\langle x', x\rangle_X = x'(x)$.

Let V be a Hilbert space. An operator $P : V \to V'$ will be called *symmetric* if

$$\langle Pv, w\rangle_V = \langle Pw, v\rangle_V \quad \text{for all } v, w \in V,$$

and *non–negative definite* if

$$\langle Pv, v\rangle_V \geq 0 \quad \text{for all } v \in V.$$

It is easy to check that a symmetric operator must be linear and, moreover, continuous by the uniform boundedness theorem.

In this section we study the evolution equation

$$Pu_{tt} + Q(t)u_t + A(t, u) = F(t, u), \qquad t \in J, \tag{2.1}$$

where P is symmetric and non–negative definite from V into V'. We suppose that the *dissipation operator* $Q(t)$ is, for each $t \in J$, symmetric and non–negative definite from an appropriate Hilbert space Y into its dual Y'. In addition, assume $Q \in C(J \to B(Y, Y'))$, that is $\langle Q(\cdot)v, w\rangle_Y : J \to \mathbb{R}$ is continuous for each $v, w \in Y$. Note that $P \equiv 0$ and $Q \equiv 0$ are specifically allowed.

Finally, the operators A and F are given by

$$A : J \times W \to W', \qquad F : J \times X \to X',$$

with W, X Banach spaces and W', X' their duals. In order to define the energy $\mathcal{E}u$ of a solution of (2.1), see below, it is necessary that there exist C^1 potentials

$$\mathcal{A} : J \times W \to \mathbb{R}, \qquad \mathcal{F} : J \times X \to \mathbb{R},$$

such that for each fixed t the operators A and F are the Fréchet derivatives with respect to u of $\mathcal{A}$ and $\mathcal{F}$, respectively; by normalization we can take $\mathcal{A}(t, 0) \equiv 0$, $\mathcal{F}(t, 0) \equiv 0$.

Now suppose that there is given a nontrivial subspace G of V, W, X and Y – not necessarily closed. Let

$$K = \{\varphi : J \to G \mid \varphi \in C(J \to W) \cap C(J \to X) \cap C^1(J \to V) \cap AC(J \to Y)\}.$$

We say that u is a (*strong*) *solution* of (2.1) if

(a) $u \in K$;

(b) Distribution Identity:

$$\langle Pu_t(\tau), \varphi(\tau)\rangle_V \Big|_0^t = \int_0^t \{\langle Pu_t(\tau), \varphi_t(\tau)\rangle_V - \langle Q(\tau)u_t(\tau), \varphi(\tau)\rangle_Y - \langle A(\tau, u(\tau)), \varphi(\tau)\rangle_W + \langle F(\tau, u(\tau)), \varphi(\tau)\rangle_X\} d\tau$$

for all $t \in J$ and $\varphi \in K$;

(c) Energy Conservation:

$$\mathcal{E}u(t) - \mathcal{E}u(0) \le -\int_0^t \{\langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y - \mathcal{A}_t(\tau, u(\tau)) + \mathcal{F}_t(\tau, u(\tau))\} d\tau,$$

where

$$\mathcal{E}u(t) = \tfrac{1}{2}\langle Pu_t(t), u_t(t)\rangle_V + \mathcal{A}(t, u(t)) - \mathcal{F}(t, u(t)), \quad t \in J, \tag{2.2}$$

is the total energy of u.

For our first result we assume even more that $Q \in C^1(J \to B(Y, Y'))$ with $Q_t(t) : Y \to Y'$ being non–positive definite and (necessarily) symmetric for all $t \in J$.

Theorem 1. *Suppose there is a constant $q > 2$ such that, for all $(t, u) \in J \times G$ for which $\mathcal{A}(t, u) < \mathcal{F}(t, u)$, we have*

$$\langle A(t, u), u\rangle_W - \langle F(t, u), u\rangle_X \le q\{\mathcal{A}(t, u) - \mathcal{F}(t, u)\} \tag{2.3}$$

and

$$\mathcal{A}_t(t, u) - \mathcal{F}_t(t, u) \le 0. \tag{2.4}$$

Then there is no solution u of (2.1) *on J such that $\mathcal{E}u(0) < 0$.*

Proof. We first show that, along any solution u of (2.1) with $\mathcal{E}u(0) < 0$, we have

$$\mathcal{A}(t, u(t)) - \mathcal{F}(t, u(t)) < 0 \tag{2.5}$$

and

$$\mathcal{E}u(t) - \mathcal{E}u(0) \le -\int_0^t \langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y d\tau \tag{2.6}$$

for all $t \in J$. Indeed, since $\mathcal{E}u(0) < 0$, by (2.2) and the fact that P is a non–negative definite operator, we obtain

$$\mathcal{A}(0, u(0)) - \mathcal{F}(0, u(0)) < 0.$$

If (2.5) fails to hold for all $t \in J$, then there exists a first value $T > 0$ such that

$$\mathcal{A}(T, u(T)) - \mathcal{F}(T, u(T)) = 0.$$

Consequently by (2.4)

$$\mathcal{A}_t(t, u(t)) - \mathcal{F}_t(t, u(t)) \leq 0 \quad \text{for all } t \in [0, T).$$

Then by the energy conservation law (c) we get, since $Q(t)$ is a non–negative definite operator,

$$\mathcal{E}u(t) \leq \mathcal{E}u(0) < 0 \quad \text{in } [0, T].$$

Hence by (2.2)

$$\mathcal{A}(T, u(T)) - \mathcal{F}(T, u(T)) < 0,$$

which is the required contradiction. Consequently (2.5) holds, and in turn by (c) and (2.4) also (2.6) is verified.

The inequalities (2.5) and (2.6) being shown, we now define, corresponding to any solution u of (2.1) on J,

$$\begin{aligned} \mathcal{I}(t) = \langle Pu(t), u(t)\rangle_V + \int_0^t \{\langle Q(\tau)u(\tau), u(\tau)\rangle_Y + (\tau - t)\langle Q_t(\tau)u(\tau), u(\tau)\rangle_Y\} d\tau \\ + (T_0 - t)\langle Q(0)u(0), u(0)\rangle_Y + \beta(t + t_0)^2, \end{aligned} \tag{2.7}$$

where t_0, T_0, β are positive constants which will be fixed later (see [3, p.145]). Then one finds, from the assumption that P, $Q(t)$, $Q_t(t)$ are linear, continuous and symmetric,

$$\begin{aligned} \mathcal{I}'(t) &= 2\langle Pu(t), u_t(t)\rangle_V + \langle Q(t)u(t), u(t)\rangle_Y - \langle Q(0)u(0), u(0)\rangle_Y \\ &\quad - \int_0^t \langle Q_t(\tau)u(\tau), u(\tau)\rangle_Y + 2\beta(t + t_0) \\ &= 2\langle Pu(t), u_t(t)\rangle_V + 2\int_0^t \langle Q(\tau)u(\tau), u_t(\tau)\rangle_Y d\tau + 2\beta(t + t_0). \end{aligned}$$

With the help of the distribution identity (b), with $\varphi = u \in K$, it follows next that

$$\begin{aligned} \tfrac{1}{2}\mathcal{I}''(t) &= \{\langle Pu_t(t), u_t(t)\rangle_V - \langle Q(t)u_t(t), u(t)\rangle_Y - \langle A(t, u(t)), u(t)\rangle_W + \langle F(t, u(t)), u(t)\rangle_X\} \\ &\qquad + \langle Q(t)u(t), u_t(t)\rangle_Y + \beta \\ &= \langle Pu_t(t), u_t(t)\rangle_V - \langle A(t, u(t)), u(t)\rangle_W + \langle F(t, u(t)), u(t)\rangle_X + \beta. \end{aligned}$$

This may be simplified by using (2.3) and (2.2), namely

$$\begin{aligned} \tfrac{1}{2}\mathcal{I}''(t) &\geq \langle Pu_t(t), u_t(t)\rangle_V - q\{\mathcal{A}(t, u(t)) - \mathcal{F}(t, u(t))\} + \beta \\ &= \left(1 + \frac{q}{2}\right)\langle Pu_t(t), u_t(t)\rangle_V - q\mathcal{E}u(t) + \beta. \end{aligned}$$

Now assume for contradiction that u has the property $\mathcal{E}u(0) < 0$. Using (2.6) to eliminate $\mathcal{E}u(t)$ from the previous inequality and choosing $\beta = 2|\mathcal{E}u(0)|$, we obtain the crucial estimate

$$\mathcal{I}''(t) \geq (q+2)\{\langle Pu_t(t), u_t(t)\rangle_V + \beta\} + 2q\int_0^t \langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y d\tau. \tag{2.8}$$

Take t_0 so large that $\mathcal{I}'(0) = 2\langle Pu(0), u_t(0)\rangle_V + 2\beta t_0 > 0$. Then, using the fact that P, $Q(t)$ are non–negative definite yields

$$\mathcal{I}'',\ \mathcal{I}',\ \mathcal{I} > 0 \qquad \text{on } J.$$

We assert that

$$\mathcal{I}\mathcal{I}'' - \alpha\mathcal{I}^{'2} \geq 0 \quad \text{on } [0, T_0], \tag{2.9}$$

where $\alpha = (q+2)/4$. Indeed, put

$$\mathbb{A} = \langle Pu(t), u(t)\rangle_V + \int_0^t \langle Q(\tau)u(\tau), u(\tau)\rangle_Y d\tau + \beta(t+t_0)^2,$$
$$\mathbb{C} = \langle Pu_t(t), u_t(t)\rangle_V + \int_0^t \langle Q(\tau)u_t(\tau), u_t(\tau)\rangle_Y d\tau + \beta,$$

and $\mathbb{B} = \frac{1}{2}\mathcal{I}'$. Since $Q(t)$ is non–negative definite and $Q_t(t)$ non–positive definite for each $t \in J$, we see that

$$\mathbb{A} \leq \mathcal{I} \qquad \text{on } [0, T_0]. \tag{2.10}$$

Moreover, by (2.8) and the fact that $2q > q+2$,

$$\mathbb{C} \leq \mathcal{I}''/(q+2) \qquad \text{on } J. \tag{2.11}$$

Now observe that, for all $(\xi, \eta) \in \mathbb{R}^2$ and $t \in J$,

$$\begin{aligned}\mathbb{A}\xi^2 + 2\mathbb{B}\xi\eta + \mathbb{C}\eta^2 = {} & \langle \xi Pu(t) + \eta Pu_t(t), \xi u(t) + \eta u_t(t)\rangle_V \\ & + \int_0^t \langle \xi Q(\tau)u(\tau) + \eta Q(\tau)u_t(\tau), \xi u(\tau) + \eta u_t(\tau)\rangle_Y d\tau \\ & + \beta\{(t+t_0)\xi + \eta\}^2 \geq 0,\end{aligned}$$

because P, $Q(t)$ are linear, symmetric and non–negative definite. Thus $\mathbb{A}\mathbb{C} - \mathbb{B}^2 \geq 0$. In turn (2.9) is valid by virtue of (2.10), (2.11) and the fact that $\mathbb{A}$, $\mathbb{C} > 0$.

Of course $\alpha > 1$ since $q > 2$ by assumption. The inequality (2.9) can be written as $(\mathcal{I}^{-\alpha}\mathcal{I}')' \geq 0$, so

$$\frac{\mathcal{I}'(t)}{\mathcal{I}^\alpha(t)} \geq \frac{\mathcal{I}'(0)}{\mathcal{I}^\alpha(0)} > 0 \quad \text{for } t \in [0, T_0].$$

This is a Riccati inequality with blow–up time

$$T < \frac{1}{\alpha - 1} \frac{\mathcal{I}(0)}{\mathcal{I}'(0)}.$$

Consequently, if T_0 is chosen as the right hand side of the above inequality, we have a contradiction. In fact, since $\mathcal{I}(0)$ depends on T_0, this gives an easily solved linear equation for T_0, the solution being *positive* for all t_0 large enough, e.g., whenever

$$\beta t_0 > \frac{2}{q-2} \langle Q(0)u(0), u(0)\rangle_Y - \langle Pu(0), u_t(0)\rangle_V.$$

(An optimal choice for t_0, to minimize T_0, is easily determined, but is unnecessary for our purposes.) This completes the proof of the theorem.

Corollary. *Assume that $\mathcal{A}(t, u) \geq 0$ on the entire space $J \times G$. Then the assertion of Theorem 1 remains true if* (2.3) *is replaced by the property*

(2.12) $\qquad (t, u) \in J \times G$ *and* $\mathcal{F}(t, u) > 0$ *implies* $\mathcal{F}_t(t, u) \geq 0$ *and* $(\mathcal{A}/\mathcal{F})_t(t, u) \leq 0$.

Proof. We show that (2.12) implies (2.4). Thus let $(t, u) \in J \times G$ be such that $\mathcal{A}(t, u) < \mathcal{F}(t, u)$. Then $\mathcal{F}(t, u) > 0$ and so (2.12) can be applied, that is

$$\mathcal{A}_t(t, u) - \mathcal{F}_t(t, u) \leq \frac{\mathcal{F}_t(t, u)}{\mathcal{F}(t, u)} \{\mathcal{A}(t, u) - \mathcal{F}(t, u)\} \leq 0.$$

Special Case. $\mathcal{A}(t, u) = a(t)\hat{\mathcal{A}}(u)$ and $\mathcal{F}(t, u) = b(t)\hat{\mathcal{F}}(u)$. If $a \geq 0$ and $b > 0$ on J, and $\hat{\mathcal{A}} \geq 0$ on G, then (2.12) holds provided a/b is non–increasing and b is non–decreasing on J.

Remark. The special case of (2.1) when $F = F(u)$, $Q(t) \equiv 0$, $A(t, u) = A(t)u$ is a linear non–negative definite symmetric operator in W, with W a Hilbert space, was treated in [2]. In this case, condition (2.4) is implied by condition (A-iii) of Theorem 1 of [2]. Indeed, we clearly have

$$\mathcal{A}(t, u) = \int_0^1 \langle \tau A(t)u, u\rangle_W d\tau \geq 0$$

and moreover

$$\mathcal{A}_t(t, u) = \int_0^1 \langle \tau A_t(t)u, u\rangle_W d\tau \leq 0$$

by (A-iii). Hence $\mathcal{A}_t(t,u) - \mathcal{F}_t(t,u) = \mathcal{A}_t(t,u) \le 0$ for all $(t,u) \in J \times W$, as required.

A similar remark applies to the parabolic subcase of (2.1) treated in Theorem 1 of [1], for which $P \equiv 0$ and the corresponding space V is no longer needed.

§3. The case with general dissipation.

Here we study the more general evolution equation

$$Pu_{tt} + Q(t,u,u_t) + A(t,u) = F(t,u), \qquad t \in J, \tag{3.1}$$

where P, A, F are as in Section 2, while Q is a nonlinear damping operator. We introduce the set

$$S = J \times X \times Y$$

and suppose $Q \in C(S \to X')$. Here Y no longer need be a Hilbert space.

By a (*strong*) *solution* of (3.1) we mean a function u such that

(a) $u \in K$;
(b) Distribution Identity:

$$\langle Pu_t(\tau), \varphi(\tau)\rangle_V \Big|_0^t = \int_0^t \{\langle Pu_t(\tau), \varphi_t(\tau)\rangle_V - \langle Q(\tau, u(\tau), u_t(\tau)), \varphi(\tau)\rangle_X - \langle A(\tau, u(\tau)), \varphi(\tau)\rangle_W + \langle F(\tau, u(\tau)), \varphi(\tau)\rangle_X\} d\tau$$

for all $t \in J$ and $\varphi \in K$;
(c) Energy Conservation:

$$\mathcal{E}u(t) - \mathcal{E}u(0) \le -\int_0^t \{\mathcal{D}(\tau, u(\tau), u_t(\tau)) - \mathcal{A}_t(\tau, u(\tau)) + \mathcal{F}_t(\tau, u(\tau))\} d\tau,$$

where $\mathcal{E}u$ is defined as in (2.2).

The function $\mathcal{D} = \mathcal{D}(t,u,v) \in C(S \to \mathbb{R}_0^+)$ is the *dissipation rate*. Clearly it has the property that

$$\mathcal{D}(\cdot, u(\cdot), u_t(\cdot)) \in L^1_{\mathrm{loc}}(J) \tag{3.2}$$

along any solution u of (3.1).

We assume that *there is an exponent* $m > 1$, *a real constant* κ *and a positive function* $\delta = \delta(t)$ *of class of* $C^1(J)$ *such that*

$$\|Q(t,u,v)\|_{X'} \le [\delta(t) \cdot \|u\|_X^\kappa]^{1/m} [\mathcal{D}(t,u,v)]^{1/m'} \quad \text{on } S,$$

where m' *is the Hőlder conjugate of* m.

For a complete discussion of this abstract formulation, see [5] and also [6, 7].

We now present a generalization of Theorem 1 of [5], covering the case when A and F depend on t as well as u. In this result, condition (2.3) is replaced by the following three growth conditions:

There exists $q > 0$ *such that*

$$\langle A(t,u),u\rangle_W \le q\mathcal{A}(t,u) \quad \textit{for all } (t,u) \in J \times G. \tag{3.3}$$

For all $\varepsilon > 0$ *there are there are positive constants* $c_1 = c_1(\varepsilon)$, $c_2 = c_2(\varepsilon)$ *and an exponent* $p > 1$ *such that*

$$c_1\mathcal{F}(t,u) \le c_2\|u\|_X^{\,p} \le \langle F(t,u),u\rangle_X - q\mathcal{F}(t,u) \tag{3.4}$$

whenever $(t,u) \in J \times G$ *and* $\mathcal{F}(t,u) \ge \varepsilon$.

There exists a positive constant c_3 *such that*

$$\|Pv\|_{V'}^2 \le c_3\langle Pv,v\rangle_V \qquad \text{for all } v \in V. \tag{3.5}$$

Finally, we suppose that the space X is continuously embedded in V.

Theorem 2. *Suppose* $\delta'(t) = o(\delta(t))$ *as* $t \to \infty$. *Assume that* $p > \max\{2, m+\kappa\}$ *and*

$$\min\{1, \delta^{(1+\theta)/(1-m)}\} \notin L^1(J),$$

for some constant θ, *with* $0 < \theta < \theta_0$, *where*

$$\theta_0 = \begin{cases} \dfrac{p-m-\kappa}{pm-(p-m-\kappa)} & \textit{if} \quad \dfrac{\kappa}{p} + \dfrac{m}{2} \ge 1 \\[2ex] \dfrac{p-2}{2p-(p-2)} & \textit{if} \quad \dfrac{\kappa}{p} + \dfrac{m}{2} < 1. \end{cases}$$

Then no solution u *of* (3.1) *can exist on* J *whenever* (2.4) *holds and* $\mathcal{E}u(0) < 0$.

Proof. We first observe, in view of (2.4), that along any solution u of (3.1) the conditions (2.5) and

$$\mathcal{E}u(t) - \mathcal{E}u(0) \le -\int_0^t \mathcal{D}(\tau, u(\tau), u_t(\tau))d\tau \tag{3.6}$$

hold on J, see the proof of Theorem 1 above. The proof of the theorem therefore reduces exactly to that of Theorem 1 of [5], or more precisely to the case where $k(t) = 1$, $\rho = \delta^{1/(m-1)}$ on J, and P is linear – these latter specializations are not essential, but have been made only for the sake of simplicity. Note also that the Remark at the end of Section 3 of [5] has been used to treat the case when κ is not zero.

§4. Examples.

A number of concrete examples relative to linear operators A were given in Section III of [2], to which we refer the reader. Example VI of [2, p.16] in particular deserves special mention since for precision the space Y as well as W must be chosen as $H_0^1(\Omega)$.

Other concrete operators $A(u)$, not appearing in [2], are noted in [4] and in Section 6 of [6], notably the degenerate Laplacian $-\mathrm{div}(|Du|^{p-2}Du)$ and the polyharmonic operator $(-\Delta)^L$, where $L \geq 1$ is an integer, and still further examples are given in Section 4 of [5].

We conclude with a discussion of several model cases, the first being

$$u_{tt} - \Delta u = f(t, x, u), \quad x \in \Omega,\ t \in J, \tag{4.1}$$

where

$$f(t, x, u) = g(t, x)u + c|u|^{p-2}u, \quad c > 0,\ p > 2.$$

For the conclusions below we ask that

$$\frac{\partial g}{\partial t} \geq 0 \quad \text{on } J \times \Omega, \qquad \|g(t, \cdot)\|_{L^{p/(p-2)}} \leq c_0 \quad \text{for all } t \in J. \tag{4.2}$$

The natural spaces associated to (4.1) are $V = L^2(\Omega)$, $W = H_0^1(\Omega)$, $X = L^p(\Omega)$ and $G = L^p(\Omega) \cap H_0^1(\Omega)$. (The remaining spaces S and Y are unneeded, since we have omitted the damping terms from the equation. Note moreover that in applications of Theorem 2 the domain Ω must be bounded, in order that $X = L^p(\Omega)$ be embedded in $V = L^2(\Omega)$.)

The operator P corresponding to (4.1) is given by $\langle Pv, w\rangle_V = (v, w)_{L^2}$; clearly P is symmetric and positive definite, and satisfies (3.5). Moreover $A(u) = -\Delta u$, so

$$\langle A(u), u\rangle_{H_0^1} = \|u\|^2_{H_0^1}, \qquad \mathcal{A}(u) = \frac{1}{2}\|u\|^2_{H_0^1}$$

and $q\mathcal{A}(u) \geq \langle A(u), u\rangle_{H_0^1}$ whenever $q \geq 2$. On the other hand,

$$\mathcal{F}(t, u) = \frac{1}{2}\int_\Omega g(t, x)|u|^2 dx + \frac{c}{p}\|u\|^p_{L^p}. \tag{4.3}$$

Hence, (2.4) is satisfied on $J \times G$ by $(4.2)_1$ – we assume suitable regularity of $g = g(t, x)$ so that $\mathcal{F}_t(t, u)$ can be calculated on $J \times G$ by differentiation under the integral sign in (4.3).

Moreover,

$$\langle F(t, u), u\rangle_{L^p} = \int_\Omega g(t, x)|u|^2 dx + c\|u\|^p_{L^p}.$$

Therefore (2.3) is verified on $J \times G$ and Theorem 1 can be applied, *provided that*

$$\left(\frac{q}{2} - 1\right)\int_\Omega g(t, x)|u|^2 dx \leq c\left(1 - \frac{q}{p}\right)\|u\|^p_{L^p}, \tag{4.4}$$

with $2 < q \le p$. This obviously occurs whenever $g(t,x) \le 0$ a.e. in $J \times \Omega$, but fails when $g(t,x) > 0$ on $J \times \Omega$ and $|u(x)|$ is sufficiently small.

Nevertheless non–continuation still occurs. To see this, observe from (2.2) and (3.6) that necessarily, for all solutions u of (2.1) satisfying $\mathcal{E}u(0) < 0$, we have

$$\mathcal{F}(t,u(t)) = \tfrac{1}{2}\|u_t(t)\|^2_{L^2} + \tfrac{1}{2}\|u(t)\|^2_{H^1_0} - \mathcal{E}u(t) \ge -\mathcal{E}u(t) \ge -\mathcal{E}u(0) = |\mathcal{E}u(0)|, \quad t \in J.$$

Thus we only need to verify (4.4) whenever $(t,u) \in J \times G$ is such that

$$\mathcal{F}(t,u) \ge |\mathcal{E}u(0)| = \varepsilon, \qquad \text{that is} \qquad \frac{1}{2}\int_\Omega g(t,x)|u|^2 dx + \frac{c}{p}\|u\|^p_{L^p} \ge \varepsilon.$$

In particular, this implies by Hőlder's inequality and $(4.2)_2$

$$\frac{c_0}{2}\|u\|^2_{L^p} + \frac{c}{p}\|u\|^p_{L^p} \ge \varepsilon.$$

Consequently we need to check (4.4) only when $\|u\|_{L^p} \ge \lambda(\varepsilon) = \lambda > 0$, where

$$\frac{c_0}{2}\lambda^2 + \frac{c}{p}\lambda^p = \varepsilon.$$

In fact, rather than (4.4), consider the stronger inequality

$$\left(\frac{q}{2} - 1\right) c_0\|u\|^2_{L^p} \le c\left(1 - \frac{q}{p}\right)\|u\|^p_{L^p};$$

this however holds for $\|u(t)\|_{L^p} \ge \lambda$ provided that

$$c\left(1 - \frac{q}{p}\right)\lambda^{p-2} - \left(\frac{q}{2} - 1\right) c_0 \ge 0.$$

Thus it is enough if q obeys

$$2 < q \le \frac{c\lambda^{p-2} + c_0}{2c\lambda^{p-2} + pc_0}\, 2p = q_1;$$

note that $q_1 > 2$ since $p > 2$ by assumption. Of course $q_1 = p$ when $c_0 = 0$, while $q_1 < p$ when $c_0 > 0$ and $p > 2$.

This example can also be treated by means of Theorem 2. With the above calculations in mind, we take $q = 2$. Then (2.4) is verified as above, while (3.4) becomes

$$c_1\left(\frac{1}{2}\int_\Omega g(t,x)|u|^2 dx + \frac{c}{p}\|u\|^p_{L^p}\right) \le c_2\|u\|^p_{L^p} \le c\left(1 - \frac{2}{p}\right)\|u\|^p_{L^p}.$$

This holds if $c_2 = c(1-2/p)$ and if also the following inequality is satisfied,

$$c\left(1-\frac{2}{p}-\frac{c_1}{p}\right)\|u\|_{L^p}^p - \frac{c_0 c_1}{2}\|u\|_{L^p}^2 \geq 0.$$

As before, this is required only when $\mathcal{F}(t,u) \geq |\mathcal{E}u(0)| = \varepsilon$ on $J \times G$, i.e. when $\|u\|_{L^p} \geq \lambda(\varepsilon) = \lambda > 0$. This gives the condition

$$c_1 = \frac{2(p-2)c\lambda^{p-2}}{2c\lambda^{p-2}+pc_0}.$$

Obviously $c_1 > 0$ since $p > 2$ by assumption. Here, as in the applications of Theorem 1, it remains crucial that $c > 0$.

Next consider the problem

$$u_{tt} - \operatorname{div}(|Du|^{s-2}Du) = f(t,x,u), \quad x \in \Omega,\ t \in J,$$

where $s > 1$, $s \neq 2$, and

$$f(t,x,u) = g(t,x)|u|^{s-2}u + c|u|^{p-2}u, \quad c \geq 0,\ p > s.$$

We assume the analogue of (4.2), namely

$$\frac{\partial g}{\partial t} \geq 0 \quad \text{on } J \times \Omega, \qquad \|g(t,\cdot)\|_{L^{p/(p-s)}} \leq c_0 \quad \text{for all } t \in J.$$

Here the appropriate spaces are $V = L^2(\Omega)$, $W = W_0^{1,s}(\Omega)$, $X = L^p(\Omega)$ and $G = L^2(\Omega) \cap L^p(\Omega) \cap W_0^{1,s}(\Omega)$. Of course, as before, in applications of Theorem 2 the domain Ω must be bounded.

Using Theorem 1 when $s > 2$, and choosing $q = s$, we get non–continuation whenever $c \geq 0$. Using Theorem 2 with $q = s$, we get non–continuation for $c > 0$ and $p > \max\{2,\ s\}$.

Hence, when $s > 2$ and $c = 0$ the result given by Theorem 1 is unobtainable from Theorem 2. Conversely, when $1 < s < 2$ and $c > 0$ the result given by Theorem 2 is unobtainable from Theorem 1.

Another function f for which Theorem 1 applies, but not Theorem 2, is given by

$$f(u) = |u|^{p-1}, \quad p > 2.$$

Here

$$\langle F(u), u\rangle_{L^p} - q\mathcal{F}(u) = \left(1-\frac{q}{p}\right)\int_\Omega |u|^{p-1}u\,dx,$$

so we can take $q = p$ for Theorem 1. On the other hand, for Theorem 2 the condition

$$c_2 \|u\|_{L^p}^p \leq \left(1 - \frac{q}{p}\right) \int_\Omega |u|^{p-1} u dx$$

cannot in general be satisfied unless one assumes *a priori* that $u \geq 0$.

Theorems 1 and 2 also differ in their requirements concerning the dissipation operator Q. We leave such comparisons to the reader.

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