UNIQUENESS OF GROUND STATES FOR QUASILINEAR ELLIPTIC OPERATORS

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In a recent paper, Erbe and Tang provide a striking new identity applying to radial solutions of the quasilinear equation

$$\Delta_m u + f(u) = 0, \tag{1.1}$$

see Proposition 1 below. Here, as usual, Δ_m denotes the degenerate m-Laplace operator $\mathrm{div}(|Du|^{m-2}Du)$, $m > 1$, and $f(u)$ is a continuously differentiable function defined for $0 < u < \infty$.

The principal application in their paper is to the uniqueness of the Dirichlet problem in a ball B, with $u = 0$ on ∂B and $u > 0$ in B. Here, with the help of their identity, we consider the complementary problem of *uniqueness of radial ground states* of (1.1), that is, non-negative, non-trivial radially symmetric solutions on $\mathbb{R}^n$ such that $u = u(x) \to 0$ as $|x| \to \infty$. This problem, like that considered by Erbe and Tang, has exercised many researchers in the past decade, principally in the case when $f(u)$ is *negative* for small u and *superlinear* for large u; see in particular [CL], [Co], [CEF1], [Kw], [M], [MS], and most recently [CEF2], when $m = 2,\ n \geq 3$; and [C2] when $1 < m \leq 2$. Other non-linearities, having *sublinear* behavior for large u, have been treated in [PS1], [PS2] and [FLS].

In this paper we shall consider the general case when $1 < m < \infty$, $2 \leq n < \infty$, see Theorem 1 of Section 2; in fact the dimension n can be treated as a real parameter $n > 1$, c.f. the remark following equation (2.1).

Our results apply in particular to the important canonical nonlinearity

$$f(u) = -u^p + u^q, \tag{1.2}$$

where

$$0 < p < q < \sigma \tag{1.3}$$

and σ is the Sobolev exponent

$$\sigma = \frac{(m-1)n+m}{n-m} \quad \text{if } m < n; \qquad \sigma = \infty \quad \text{if } m \geq n. \tag{1.4}$$

Previous results for this nonlinearity have been confined for the most part to various superlinear cases with $p = 1$ and $m \leq 2$, as noted above, and to the "sublinear" case $q \leq m-1$ when $m \neq 2$. For a description of the range of exponents p, q which are explicitly covered here, see Theorems 2, 2′ below; Theorem 2 in particular shows that, for $\sigma > 1$, uniqueness always holds for the subset of (1.3) where $p \leq \sigma - 1$.

Corresponding work on the existence of ground states of (1.1) for the superlinear case can be found in [C1] and for the sublinear case in [FLS]. Moreover for the Laplace operator Kaper and Kwong have considered nonlinearities of the form (1.2) with $0 \leq p < q \leq 1$.

The above results also extend to more general equations than (1.1), namely those of the form

$$\operatorname{div}(A(|Du|)Du) + f(u) = 0, \tag{1.5}$$

where $A \in C^1(0,\infty)$. This extension is the second principal goal of the paper.

For $\rho > 0$ define

$$\Omega(\rho) = \rho A(\rho), \qquad G(\rho) = \int_0^\rho \Omega(\rho) d\rho.$$ [1]

We suppose only the following conditions on the operator A:

(1) $\Omega'(\rho) > 0$ for $\rho > 0$; $\quad \Omega(\rho) \to 0$ as $\rho \to 0$,

(2) $\Omega(\rho) \leq \text{Const.}\, \Omega'(\rho)$ for ρ near 0.

(3) $\dfrac{\rho\Omega(\rho)}{G(\rho)}$ is (non-strictly) increasing for $\rho > 0$,

Note from (1) that A, Ω, G are positive for $\rho > 0$ (the integral for G being well defined since Ω is bounded for ρ near 0). Moreover, since Ω, Ω' (i.e. G', G'') are positive for

[1] It is interesting that (1.5) is then the Euler-Lagrange equation for the variational problem

$$\delta \int \{G(|Du|) - F(u)\} dx = 0,$$

where $F(u) = \int_0^u f(s) ds$.

$\rho > 0$, the quasilinear equation (1.5) is elliptic when $Du \neq 0$ though possibly degenerate at $Du = 0$.

For the operator A, we define the critical constant

$$m = \inf_{\rho>0} \frac{\rho\Omega(\rho)}{G(\rho)} = \lim_{\rho\to 0} \frac{\rho\Omega(\rho)}{G(\rho)}. \tag{1.6}$$

It is obvious that $G(\rho) < \rho\Omega(\rho)$ for $\rho > 0$, since Ω is an increasing function. Hence $m \geq 1$.

With the assumptions (1), (2), (3) on the operator A, Theorems 1, 2, 2′ then apply equally to the more general equation (1.5), with the constant σ in Theorems 2, 2′ being given exactly as before in terms of m and n.

We note that conditions (1)–(3) hold, for example, for the operators

$$\begin{aligned}
&A(\rho) = \rho^{m-2} && (m > 1, \text{ as in } (1.1))\\
&A(\rho) = \rho^{m-2}(1 + c\rho^{\alpha}), && \alpha > 0,\ c > 0,\ m > 1,\\
&A(\rho) = \rho^{m-2}(m + \rho)e^{\rho}, && m > 1,\\
&A(\rho) = \frac{1}{\rho \log\frac{1}{\rho}}\left(1 + \frac{1}{\log\frac{1}{\rho}}\right), && 0 < \rho < 1,
\end{aligned}$$

and so forth. For the last example one finds $m = 1$, showing that all values $m \geq 1$ are possible.[2]

In this regard, it is shown in Section 2 that condition (2) is automatically satisfied whenever $m > 1$. Thus, except for the anomalous case $m = 1$, this somewhat arbitrary appearing condition can be dropped.

In Section 6 we extend the exponent range (1.3) to the set

$$-1 < p < q < \sigma, \tag{1.7}$$

the conclusions for this case being given in Theorems 3, 4 and 4′. These results are new even in the classical case of the Laplace operator. In particular, when $\sigma > 1$ uniqueness holds in the subset of (1.7) where $p \leq \sigma - 1$, $q \geq 0$.

[2]One checks that $\Omega(\rho) = \frac{1}{\log\frac{1}{\rho}}\left(1 + \frac{1}{\log\frac{1}{\rho}}\right)$, so that $\Omega' > 0$ for $\rho > 0$ while also $\Omega \to 0$, $\Omega'/\Omega \to \infty$ as $\rho \to 0$. Moreover

$$G = \rho/\log\frac{1}{\rho}, \quad \frac{\rho\Omega}{G} = 1 + 1/\log\frac{1}{\rho};$$

hence $m = 1$ and $\rho\Omega/G$ is increasing.

It should be noted in this connection that when $n > m$ and $q \geq \sigma$ there are no solutions of either (1.1) or (1.5) corresponding to the canonical nonlinearity (1.2), see [NS], pages 180-181.

Remarks. We emphasize that uniqueness of radial ground states is here understood subject to an arbitrary translation of the origin. It is important to note moreover that our results apply equally to radially symmetric "single bump" compact support ground states.

As background and motivation for the study fof radial ground states of (1.1), we note that by the Gidas–Ni–Nirenberg theorem positive ground states of (1.1) corresponding to the Laplace operator ($m = 2$) are necessarily radially symmetric about some origin $O \in \mathbb{R}^n$, provided that the function $f = f(u)$ is appropriately regular near $u = 0$ and $f'(0) < 0$. A generalization of this result to *elliptic quasilinear equations* of the form (1.5), as well as to compact support ground states and to more general nonlinearities f, also holds. For details, see the forthcoming paper [SZ].

The methods developed here for the study of the ground state problem can be applied also for the *exterior Neumann problem.* In particular, consider non-negative, non-trivial radial solutions of (1.5) in the exterior of the ball B_0 of radius R_0, subject to the boundary conditions

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B_0, \qquad u(r) \to 0 \quad \text{as } r \to \infty.$$

The question of uniqueness of solutions of this problem is treated in Section 7, the results being exactly analogous to those described earlier for the ground state problem.

2. Main results.

We consider radial ground states of the problem

$$\operatorname{div}(A(|Du|)Du) + f(u) = 0, \qquad x \in \mathbb{R}^n, \tag{1.5}$$

namely radial distribution solutions of class $C^1(\mathbb{R}^n)$ such that $u \geq 0$, $u \not\equiv 0$, $u(x) \to 0$ as $|x| \to \infty$. The operator A is assumed, without further mention, to obey conditions (1)–(3) given in the introduction, while the function $f(u)$, $0 \leq u < \infty$, satisfies the following assumptions:

(a) f is continuous on $[0, \infty)$, and $f(0) = 0$;

(b) f is continuously differentiable on $(0, \infty)$;

(c) there exists $a > 0$ such that $f(a) = 0$ and

$$f(u) < 0 \qquad \text{for } 0 < u < a,$$
$$f(u) > 0 \qquad \text{for } a < u < \infty.$$

Conditions (a) and (b) can be weakened somewhat, at the expense of a slightly more sophisticated treatment, see Section 6.

We observe that radial solutions $u = u(r)$ of (1.5) satisfy the ordinary differential equation

$$(A(|u'|)u')' + \frac{n-1}{r}A(|u'|)u' + f(u) = 0$$

for $r > 0$, or equivalently

$$[r^{n-1}A(|u'|)u']' + r^{n-1}f(u) = 0; \tag{2.1}$$

of course also $u'(0) = 0$. The expression $r^{n-1}A(|u'|)u'$ is then of class $C^1[0, \infty)$, see Section 1.1 of [FLS]. Finally, n naturally can be considered a real parameter, with $n > 1$.

We shall typically be dealing with solutions $u(r)$ of (2.1) having the property $u'(r) \le 0$. Setting $\rho = \rho(r) = |u'(r)|$, equation (2.1) can then be written

$$[r^{n-1}\Omega(\rho)]' = r^{n-1}f(u). \tag{2.2}$$

Let $H(\rho)$ be the partial Legendre transform of $G(\rho)$, namely

$$H(\rho) = \rho\Omega(\rho) - G(\rho), \qquad \rho \ge 0,$$

where it is convenient in what follows to define $\Omega(0) = G(0) = 0$ Then by Lemma 1.1.2 of [FLS] we have along solutions u of (2.1)

$$\frac{d}{dr}\left[H(\rho) + F(u)\right] = -(n-1)\frac{\rho\Omega(\rho)}{r}. \tag{2.3}$$

Next introduce the important function

$$P(r, u, \rho) = r^n\left[H(\rho) + F(u)\right] - nr^{n-1}\Omega(\rho)K(u), \tag{2.4}$$

defined for $r \geq 0$, $\rho \geq 0$, $u > 0$ ($\neq a$), where

$$K(u) = F(u)/f(u) \quad \text{and} \quad F(u) = \int_0^u f(\tau)d\tau.$$

The following proposition is due to Erbe and Tang in the special case when $A(\rho) = \rho^{m-2}$ (degenerate Laplace operator).

Proposition 1. *Let $u = u(r)$ be a non-negative solution of* (2.1) *with $u'(r) \leq 0$. Then, putting $\rho = \rho(r) = |u'(r)|$, we have*

$$\frac{d}{dr}P(r, u(r), \rho(r)) = nr^{n-1}\rho\Omega(\rho)\left\{K'(u) - \frac{G(\rho)}{\rho\Omega(\rho)} + \frac{1}{n}\right\}$$

for all $r > 0$ such that $u(r) \neq 0, a$. [Here $K'(u)$ means $dK(u)/du$.]

Proof. By (2.2) and (2.3) we have

$$\begin{aligned}\frac{d}{dr}P(r, u(r), \rho(r)) = r^{n-1}\left\{n\left[H(\rho) + F(u)\right] - (n-1)\rho\Omega(\rho)\right\} \\ - nr^{n-1}\Omega(\rho)K'(u)u' - nr^{n-1}f(u)K(u).\end{aligned}$$

Simplification of this gives Proposition 1.

Note that the identity in Proposition 1 also holds when $u'(r) > 0$, provided P is defined instead as $r^n[H(\rho) + F(u)] + nr^{n-1}\Omega(\rho)K(u)$.

Theorem 1. *Equation* (1.1)*, and more generally equation* (1.5) *with m defined by* (1.6)*, admits at most one radial ground state if conditions* (a), (b), (c) *hold and*

$$\frac{d}{du}\left[\frac{F(u)}{f(u)}\right] \geq \frac{n-m}{nm} \qquad \text{for} \quad u > 0,\ u \neq a. \tag{2.5}$$

Remarks. As noted in the introduction, uniqueness of ground states is here understood subject to arbitrary translations of the origin. When $1 \leq m < n$, the right side of (2.5) is the reciprocal of the Sobolev exponent for the space $W^{1,m}(\mathbb{R}^n)$.

Theorem 2. *Let $\sigma > 1$, where σ is defined by* (1.4)*. Equation* (1.1)*, and equally equation* (1.5)*, then admits at most one radial ground state when*

$$f(u) = -u^p + u^q \tag{2.6}$$

and

$$0 < p \le \sigma - 1, \qquad p < q < \sigma. \tag{2.7}$$

Moreover there exists $p_0 = p_0(\sigma) \in (\sigma - 1, \sigma)$ *and* $q_0 = q_0(p, \sigma) \in (\sigma - 1, \sigma)$ *such that the same conclusion holds in the additional range*

$$\sigma - 1 < p < p_0(\sigma), \qquad p < q \le q_0(p, \sigma) \tag{2.8}$$

(p_0, q_0 are respectively solutions of quadratic and cubic equations, with coefficients depending respectively on σ, and on p, σ).

Recall that when $n \le m$ the value $\sigma = \infty$ is to be used, with the natural interpretation that (2.7) is replaced by $0 < p < q < \infty$ while (2.8) is null. In fact, in this case one can even allow exponential growth to the function $f(u)$ as $u \to \infty$. We shall discuss this possibility in a forthcoming paper [PS].

When
$$m > \max\left\{1, \frac{2n}{n+2}\right\}$$
(and in particular when $m \ge 2$ or $n \le 2$), it is easy to check that $\sigma > 1$. On the other hand, for $1 < m < 2$ and for appropriate dimensions $n > 2$ it can happen that $\sigma < 1$, and even that σ is arbitrarily near zero. For these cases, the following result complements Theorem 2.

Theorem 2′. *Let* $1/3 < \sigma \le 1$. *There exist* $p_0 = p_0(\sigma) \in (0, \sigma)$ *and* $q_1 = q_1(p, \sigma) \in (0, \sigma)$ *such that equations* (1.1) *and* (1.5) *admit at most one ground state when* f *is given by* (2.4) *and*

$$0 < p < p_0, \qquad p < q \le q_1.$$

It is not known whether uniqueness holds in the entire range $0 < p < q < \sigma$, though when $m = 2$ and $q > 1$ a related result appears in recent work of Cortázar, Elgueta and Felmer [CEF2].

We conclude the section with a simple lemma showing that condition (2) is not needed when $m > 1$.

Lemma. *Suppose* $m > 1$. *Then*

$$\Omega'(\rho) \geq \frac{m-1}{\rho}\,\Omega(\rho), \qquad \rho > 0.$$

Proof. By (3) and the fact that $A \in C^1(0,\infty)$, we have for $\rho > 0$

$$0 \leq \left[\frac{\rho\Omega(\rho)}{G(\rho)}\right]' = \frac{[\rho\Omega'(\rho) + \Omega(\rho)]G(\rho) - \rho\Omega(\rho)^2}{G(\rho)^2}.$$

Hence

$$\Omega'(\rho) \geq \frac{1}{\rho}\left[\frac{\rho\Omega(\rho)}{G(\rho)} - 1\right]\Omega(\rho) \geq \frac{m-1}{\rho}\Omega(\rho)$$

by (1.6), as required.

3. Proof of Theorem 1. Part I.

We assume throughout, without further mention, that (1), (2), (3) and (a), (b), (c) hold. It is of course enough to treat the case of equation (1.5).

In the course of the proof we shall use various results of [FLS], this being allowable since the functions $A(\rho)$ and $f(u)$ here satisfy the general conditions required in [FLS], namely that f is continuous on $[0,\infty)$ and $f(0) = 0$; that A is continuous on $(0,\infty)$ and $\Omega(\rho) \to 0$ as $\rho \to 0$; and Ω is strictly increasing for $\rho > 0$.

It is well known that any radial ground state $u = u(r)$ of (1.5) has the property $u'(r) < 0$ as long as $r > 0$, $u(r) > 0$; see [FLS], Proposition 1.2.6 (i), and note that $F(u) < 0$ at the *only* place, namely $u = a$, where $u > 0$ and $f(u) = 0$. In turn, one may introduce the inverse function $t = t(u)$ of $u = u(r)$, defined for $0 < u < u(0)$. Here necessarily $u(0) > a$ by [FLS], Lemma 1.2.6 (ii).

The following result is due to Erbe and Tang in the special case when $A(\rho) = \rho^{m-2}$.

Lemma 3.1. *Let* u_1, u_2 *be two radial ground states of* (1.5), *with respective inverses* $t_1(u)$, $0 < u < \alpha_1 = u_1(0)$, *and* $t_2(u)$, $0 < u < \alpha_2 = u_2(0)$. *Put*

$$\rho_1(r) = |u_1'(r)|, \qquad \rho_2(r) = |u_2'(r)|,$$
$$\Omega_1(u) = \Omega\left(\rho_1(t_1(u))\right), \qquad \Omega_2(u) = \Omega\left(\rho_2(t_2(u))\right)$$

and[3]

$$T = T_{12}(u) = \left(\frac{t_1}{t_2}\right)^{n-1}\frac{\Omega_1(u)}{\Omega_2(u)}, \qquad 0 < u < \alpha = \min(\alpha_1,\alpha_2).$$

[3] In this formula we have written t_1, t_2 rather than $t_1(u)$, $t_2(u)$. Similar notational abbreviations will be used frequently in the sequel.

Then for $a < u < \alpha$ we have (with $' = d/du$)

$$T'_{12}(u) > 0 \quad \text{if and only if} \quad t'_2(u) < t'_1(u),$$

while for $0 < u < a$

$$T'_{12}(u) > 0 \quad \text{if and only if} \quad t'_2(u) > t'_1(u).$$

Proof. By logarithmic differentiation

$$\frac{1}{T}\frac{dT}{du} = \frac{d}{du}\log T = \frac{d}{du}\log\left(t_1^{n-1}\Omega_1(u)\right) - \frac{d}{du}\log\left(t_2^{n-1}\Omega_2(u)\right)$$

$$= \frac{1}{r^{n-1}\Omega(\rho)}\frac{d}{dr}\left[r^{n-1}\Omega(\rho)\right]\frac{dr}{du}\Bigg|_{\rho=\rho_2(r),\ r=t_2(u)}^{\rho=\rho_1(r),\ r=t_1(u)} \tag{3.1}$$

$$= \frac{1}{\rho\Omega(\rho)}\Bigg|_{\rho=\rho_1(t_1(u))}^{\rho=\rho_2(t_2(u))} \cdot f(u),$$

where in the last step we have again used the fundamental equation (2.2) and the fact that $u'(r) < 0$ when $0 < u < \alpha$.

We observe that $\rho\Omega(\rho)$ is an *increasing* function by assumption (1), and moreover that

$$\rho_2(t_2(u)) = -1/t'_2(u), \quad \rho_1(t_1(u)) = -1/t'_1(u).$$

Thus $\rho_2(t_2(u)) > \rho_1(t_1(u))$ if and only if $t'_2(u) > t'_1(u)$. Hence *dT/du has the same sign as $f(u)$ if and only if $t'_2(u) < t'_1(u)$*. This completes the proof, once we note that $f(u) > 0$ for $a < u < \alpha$ and $f(u) < 0$ for $0 < u < a$.

Lemma 3.2. *Let $u = u(r)$ be a radial ground state of* (1.5)*, and set $\rho = |u'(r)|$. Then*

$$r^{n-1}\Omega(\rho) \to \textit{finite limit } \lambda \geq 0 \tag{3.2}$$

and

$$\liminf_{r\to\infty} r^n[H(\rho) + F(u)] = 0. \tag{3.3}$$

Proof. The first result is just Lemma 3.6.1 of [FLS]. We indicate the proof. By (2.2)

$$[r^{n-1}\Omega(\rho)]' = r^{n-1}f(u(r)) \le 0 \qquad \text{when } u(r) < a.$$

Hence $r^{n-1}\Omega(\rho)$ is non-increasing when r is so large that $u(r) < a$, i.e. in the interval (r_0, ∞), where $u(r_0) = a$. This proves (3.2).

Next by (3.2) we see that $\Omega(\rho) \to 0$ as $r \to \infty$. Thus by the ellipticity condition (1) also $\rho \to 0$. Since also $u(r) \to 0$, it follows that

$$H(\rho) + F(u) \to 0 \qquad \text{as } r \to \infty.$$

By integration of (2.3) we thus obtain

$$H(\rho) + F(u) = (n-1)\int_r^\infty \frac{\rho(s)\Omega(\rho(s))}{s}ds \tag{3.4}$$

so that in particular $H(\rho) + F(u) \ge 0$ for all r. Hence

$$0 \le r^n[H(\rho) + F(u)] \le r^n H(\rho)$$

in (r_0, ∞), since $F(u) < 0$ for $u \le a$. In turn

$$\begin{aligned} 0 \le r^n H(\rho) &= r^n[\rho\Omega(\rho) - G(\rho)] \\ &\le r^n \rho\Omega(\rho) && \text{since } G \ge 0 \\ &\le (\lambda + 1) r\rho && \text{by (3.2)}, \end{aligned}$$

for r sufficiently large. But

$$\liminf_{r\to\infty} r\rho = 0.$$

In fact, otherwise,

$$\rho \ge \text{Const.}/r \qquad \text{for large } r,$$

which is impossible since $\rho \in L^1[0, \infty)$. Indeed,

$$0 \le u(r) = u(0) + \int_0^r u'(s)ds = u(0) - \int_0^r \rho(s)ds,$$

that is

$$\int_0^r \rho(s)ds \le u(0) \qquad \text{for all } r.$$

Hence (3.3) holds, completing the proof of the lemma.

Note. The above proof is easily seen to hold equally when u has compact support, that is $u(r) > 0$ for $0 \le r < b$ and $u(r) = u'(r) = 0$ when $r \ge b$. Of course in this case the limit in (3.2) is necessarily zero, and equally

$$r^n[H(\rho) + F(u)] \equiv 0 \qquad \text{for } r \ge b.$$

Lemma 3.3. *If $t_2(u) - t_1(u) > 0$ on an interval $I \subset (0, \alpha)$, then $t_2(u) - t_1(u)$ can have at most one critical point on I. If such a point occurs, it must be a strict maximum.*

Moreover, if $I = (0, d)$, $d \le \alpha$, then $t_2'(u) - t_1'(u) < 0$ on I.

The first part of this result is just Lemma 3.3.1 of [FLS]. (The conditions (F1), (F2) in that lemma are used only to guarantee the hypotheses on $F(u)$ in Lemma 1.2.6, and hence are not required here – see also Lemma 3.6.3.)

The second part is Lemma 3.6.5 of [FLS].

Lemma 3.4. *Let $u_1(r)$ and $u_2(r)$ be two different ground states for* (1.5)*. Then the graphs $u = u_1(r)$ and $u = u_2(r)$ cannot intersect at any point in the set* $0 < u \le a$, $r > 0$.

This is just Theorem 3.6.7 of [FLS], together with unique continuation (see Proposition A2 of [FLS] with the modified initial conditions $u(a) > 0$ and $u'(a) < 0$).

4. Proof of Theorem 1. Part II.

Let u_1, u_2 be two different radial ground states, with $u_1(0) = \alpha_1$, $u_2(0) = \alpha_2$. We assert that $\alpha_1 \ne \alpha_2$. Indeed, if $\alpha_1 = \alpha_2$, then as noted at the beginning of Section 3 we must have $f(\alpha_1) > 0$. By Proposition A4 of the Appendix of [FLS], together with conditions (2) and (b), (c), it then follows that $u_1(r) \equiv u_2(r)$ as long as $r \le r_a$, where $u_1(r_a) = u_2(r_a) = a$. But $u'(r) < 0$ as long as $r > 0$, $u(r) > 0$. Therefore, because equation (2.1) is singular *only* when $u'(r) = 0$, it is clear that the Cauchy problem for (2.1) at $r = r_a$ is unique, that is $u_1(r) \equiv u_2(r)$ for all $r > 0$. (This is Proposition A2 of [FLS]). This contradicts the assumption that u_1 and u_2 are different, and the assertion is proved. (An alternative method to prove the assertion appears on page 227 of [FLS].)

For definiteness we now suppose $\alpha_1 < \alpha_2$, where of course $\alpha_1 > a$. We first show that the graphs of $u_1(r)$ and $u_2(r)$ intersect at most once in the region $r \ge 0$, $u > 0$.

By Lemma 3.4 there can be no intersection at any point (R, U) where $R \geq 0$, $0 < U \leq a$. We claim on the other hand that there can be at most one intersection (R, U) where $U > a$. Suppose in fact that there were two such points, (R_I, U_I) and (R_{II}, U_{II}). Let R_I be the first intersection and R_{II} the second (intersection points must be isolated by the uniqueness of the Cauchy problem), that is $R_I < R_{II}$, $U_I > U_{II}$ and

$$u_1(R_I) = u_2(R_I) = u_I, \qquad u_1(R_{II}) = u_2(R_{II}) = u_{II}.$$

For r between 0 and R_I we have $u_1(r) < u_2(r)$, that is

$$t_2(u) > t_1(u) \quad \text{for} \quad u_I < u \leq \alpha,$$

where $\alpha = \min(\alpha_1, \alpha_2) = \alpha_1$. By Lemma 3.3 it follows that $t_2(u) - t_1(u)$ can have at most one critical point on the interval (u_I, α), which (should it occur at all) would have to be a strict maximum. But

$$t_2' - t_1' \to \infty, \qquad \text{as} \quad t \to \alpha^- \qquad (\alpha = \alpha_1)$$

(since $t_1' \to -\infty$ and $t_2' \to$ finite limit as $u \to \alpha^-$), while clearly

$$t_2' - t_1' > 0 \quad \text{at} \quad t = u_I.$$

Hence necessarily

$$t_2' - t_1' > 0 \qquad \text{for} \quad u_I \leq u < \alpha. \tag{4.1}$$

Between R_I and R_{II} the same argument shows (since $t_1 > t_2$ for $u_{II} < u < u_I$) that there is exactly one point $u = u_c$ between u_I and u_{II} such that $t_2'(u_c) = t_1'(u_c)$, and

$$\begin{aligned} t_2' - t_1' > 0 &\qquad \text{for} \quad u_c < u < u_I \\ t_2' - t_1' < 0 &\qquad \text{for} \quad u_{II} < u < u_c. \end{aligned}$$

Combining with (4.1), we have specifically

$$t_2'(u) > t_1'(u) \qquad \text{for} \quad u_c < u < \alpha; \tag{4.2}$$

of course $u_c > a$.

Let $C = T_{12}(u_c)$. By (b) we have $f(u) > 0$ for $u \geq u_c$. Then from (4.2) and Lemma 3.1 we see that $T'_{12}(u) < 0$ for $u_c < u < \alpha$. Therefore

$$C > T_{12}(u), \qquad u_c < u < \alpha. \tag{4.3}$$

Now we apply Proposition 1 (this is the idea of Erbe and Tang) to each of the solutions $u_1(r)$, $u_2(r)$. Thus we have

$$\begin{aligned} P(R_{1c}, u_c, \rho_1(R_{1c})) &= n\int_0^{R_{1c}} r^{n-1}\rho_1\Omega(\rho_1)L(u_1,\rho_1)dr \\ P(R_{2c}, u_c, \rho_2(R_{2c}) &= n\int_0^{R_{2c}} r^{n-1}\rho_2\Omega(\rho_2)L(u_2,\rho_2)dr, \end{aligned} \tag{4.4}$$

where

$$L(u,\rho) = K'(u) - \frac{G(\rho)}{\rho\Omega(\rho)} + \frac{1}{n}, \tag{4.5}$$

and

$$\rho_1 = \rho_1(r) = |u_1'(r)|, \qquad \rho_2 = \rho_2(r) = |u_2'(r)|$$

$$R_{1c} = t_1(u_c), \qquad R_{2c} = t_2(u_c).$$

Then by subtraction, and change of integration variable from r to u, we derive

$$\begin{aligned} &P(R_{1c}, u_c, \rho_1(R_{1c})) - CP(R_{2c}, u_c, \rho_2(R_{2c})) \\ &\quad = n\int_\alpha^{u_c} \{Ct_2^{n-1}\Omega_2(u)L(u,\rho_2(t_1)) - t_1^{n-1}\Omega_1(u)L(u,\rho_1(t_1))\}du \\ &\qquad + nC\int_{\alpha_2}^{\alpha} t_2^{n-1}\Omega_2(u)L(u,\rho_2(t_1))du, \end{aligned} \tag{4.6}$$

where $\Omega_1(u) = \Omega(\rho_1(r(u)))$, $\Omega_2(u) = \Omega(\rho_2(r(u)))$.

By (4.2) and the fact that $\rho = -1/t'$, we obtain $\rho_2(t_2(u)) > \rho_1(t_1(u))$ for $u_c < u < \alpha$. Hence by the main assumption (3)

$$L(u, \rho_2(t_2) \geq L(u, \rho_1(t_1)), \qquad u_c < u < \alpha.$$

It follows that

$$\begin{aligned} &\{Ct_2^{n-1}\Omega_2(u)L(u,\rho_2(t_2)) - t_1^{n-1}\Omega_1(u)L(u,\rho_1(t_1))\} \\ &\qquad \geq \{Ct_2^{n-1}\Omega_2(u) - t_1^{n-1}\Omega_1(u)\}L(u,\rho_2(t_2) \\ &\qquad = t_2^{n-1}\Omega_2(u)\{C - T_{12}(u)\}L(u,\rho_2(t_2)). \end{aligned} \tag{4.7}$$

Now from (4.5) and (1.6) it is clear that

$$L(u,\rho) \geq K'(u) - \frac{1}{m} + \frac{1}{n} \geq 0, \tag{4.8}$$

by the principal hypothesis (2.5). Hence, using (4.3), both integrands on the right hand side of (4.6) are non-negative; in turn, since $u_c < \alpha < \alpha_2$ the right hand side of (4.6) is non-positive.

On the other hand, from the definition (2.4),

$$\begin{aligned} &P(R_{1c}, u_c, \rho_1(R_{1c})) - CP(R_{2c}, u_c, \rho_2(R_{2c})) \\ &\qquad = t_1(u_c)^n \left[H(\rho_1(R_{1c})) + F(u_c)\right] - Ct_2(u_c)^n \left[H(\rho_2(R_{2c})) + F(u_c)\right] \\ &\qquad\qquad - n\{t_1(u_c)^{n-1}\Omega_1(u_c) - Ct_2(u_c)^{n-1}\Omega_2(u_c)\}K(u_c) \\ &\quad = I_1 + I_2. \end{aligned}$$

Using the fact that $t_1'(u_c) = t_2'(u_c)$, we obtain $\rho_1(R_{1c}) = \rho_2(R_{2c}),\ \Omega_1(u_c) = \Omega_2(u_c)$ and

$$Ct_2(u_c)^{n-1} = t_1(u_c)^{n-1},$$

so $I_2 = 0$. Similarly

$$I_1 = t_1(u_c)^{n-1} \cdot [t_1(u_c) - t_2(u_c)] \cdot [H(\rho_1(R_{1c}) + F(u_c)] > 0$$

since $t_1(u_c) > t_2(u_c)$ and $H(\rho) + F(u) > 0$ – see (3.4) and note that the integral is certainly positive for $0 < r \leq R_{1c}$. This contradicts the equality in (4.6) and shows that the two presumed intersection points cannot occur.

If we combine this with what has already been observed, that there are no intersection points below the line $u = a$, it follows that *the graphs of $u_1(r)$ and $u_2(r)$ can have at most one intersection point in the region $r \geq 0$, $u > 0$, and that this can occur only when $u > a$.*

In fact, there cannot even be a single such intersection. Suppose there were, say at (R, U). Then we use the second part of Lemma 3.3 to see that

$$t_1(u) > t_2(u), \qquad t_2'(u) > t_1'(u) \tag{4.9}$$

for $0 < u < U = u_1(R) = u_2(R)$. Since also, as for (4.1) above,

$$t_1(u) < t_2(u), \qquad t_2'(u) > t_1'(u)$$

for $U < u < \alpha$, we get

$$t_2'(u) > t_1'(u), \qquad 0 < u < \alpha.$$

It follows from Lemma 3.1 that

$$T_{12}'(u) < 0 \qquad \text{for} \quad a < u < \alpha$$
$$T_{12}'(u) > 0 \qquad \text{for} \quad 0 < u < a.$$

In particular

$$T_{12}(u) < T_{12}(a) = D \qquad \text{for} \quad 0 < u < \alpha, \quad u \neq a.$$

We now use the relation (4.6) again, but replacing C by D and the interval $u_c < u < \alpha$ by $\varepsilon < u < \alpha$, where $\varepsilon > 0$ is small. This gives

$$\begin{aligned} P(R_{1\varepsilon}, \varepsilon, \rho_1(R_{1\varepsilon})) - DP(R_{2\varepsilon}, \varepsilon, \rho_2(R_{2\varepsilon}) \\ = n \int_\alpha^\varepsilon \{D t_2^{n-1} \Omega_2(u) L(u, \rho_2(t_2)) - t_1^{n-1} \Omega_1(u) L(u, \rho_1(t_1))\} du \\ + nD \int_{\alpha_2}^\alpha t_2^{n-1} \Omega_2(u) L(u, \rho_2(t_2)) du, \end{aligned} \tag{4.10}$$

where $R_{1\varepsilon} = t_1(\epsilon)$, $R_{2\varepsilon} = t_2(\epsilon)$.

In writing (4.10) care must be taken to avoid the singularity of $P(r, u, \rho)$ at $u = a$. This is accomplished by replacing the full interval (ϵ, α) by the pair of integration intervals

$$(\varepsilon, a - \delta), \qquad (a + \delta, \alpha),$$

thus effectively isolating the singularity at $u = a$, and then letting $\delta \to 0$. The additional terms which thereby arise on the left hand side of (4.10) are then given by (in an obvious notation)

$$[P_1(u) - DP_2(u)]\Big|_{u=a-\delta}^{u=a+\delta}$$

which we denote by J. From the definition (2.4) of P, it is clear that

$$\lim_{\delta\to 0} J = n \lim_{\delta\to 0}\{Dt_2(u)^{n-1}\Omega_2(u) - t_1(u)^{n-1}\Omega_1(u)\}K(u)\Big|_{u=a-\delta}^{u=a+\delta} \tag{4.11}$$

since all remaining terms in J arise from functions which are continuous at $u = a$. The function on the right side of (4.11), when evaluated at $u = a - \delta$, can be rewritten

$$[t_2(a-\delta)]^{n-1}\Omega_2(a-\delta)F(a-\delta)\frac{T_{12}(a-\delta) - T_{12}(a)}{f(a-\delta)}.$$

By the mean value theorem and (3.1),

$$T_{12}(a-\delta) - T_{12}(a) = -T_{12}(a-\xi)\frac{1}{\rho\Omega(\rho)}\Big|_{\rho=\rho_1(R_{1,a-\xi})}^{\rho=\rho_2(R_{2,a-\xi})} \cdot f(a-\xi)\cdot\delta.$$

for some $\xi \in (0,\delta)$. Hence

$$\lim_{\delta\to 0}\frac{T_{12}(a-\delta) - T_{12}(a)}{f(a-\delta)} = -T_{12}(a)\frac{1}{\rho\Omega(\rho)}\Big|_{\rho=\rho_1(R_{1a})}^{\rho=\rho_2(R_{2a})} \cdot \lim_{\delta\to 0}\frac{f(a-\xi)}{f(a-\delta)}\,\delta. \tag{4.12}$$

Again by the definition of $K(u)$,

$$\frac{f(a-\xi)}{f(a-\delta)} = \frac{K(a-\delta)}{K(a-\xi)}\cdot\frac{F(a-\xi)}{F(a-\delta)}.$$

But since $K'(u)$ is bounded below by virtue of (2.5), we have

$$K(a-\delta) \le K(a-\xi) + \text{Const.}\,\delta.$$

Clearly $F(a) < 0$, so $K(a-\delta) \to \infty$ as $\delta \to 0$. Thus

$$0 < \frac{K(a-\delta)}{K(a-\xi)} \le 1 + \frac{\text{Const.}\,\delta}{K(a-\xi)} < 2$$

for δ suitably small. Consequently, since F is continuous at a,

$$\lim_{\delta\to 0}\frac{f(a-\xi)}{f(a-\delta)}\delta = \lim_{\delta\to 0}\frac{K(a-\delta)}{K(a-\xi)}\delta = 0. \tag{4.13}$$

The same argument yields as well (since $f > 0,\ F < 0,\ K < 0$ for $a < u < a+\delta$ and δ small) that

$$\lim_{\delta\to 0}\frac{f(a+\xi)}{f(a+\delta)}\delta = 0. \tag{4.14}$$

Thus from (4.11)–(4.14) we see without difficulty that

$$\lim_{\delta\to 0} J = 0.$$

That is, the singularity at $u = a$ is removable and formula (4.10) holds as written, naturally with the proviso that the first integral on the right side should be considered as a limit with $\delta \to 0$ (in fact it is not hard to see that this integral also exists as written, irrespective of the singularity at $u = a$).

The integrands on the right side of (4.10) are clearly non-negative, as before. Now consider the limit of the left hand side as $\varepsilon \to 0$. Since K' is bounded below by the real number $(n-m)/nm$, and $K(u) > 0$ for $0 < u < a$, we have

$$\lim_{u\to 0} K(u) = \tilde{\kappa} \geq 0. \tag{4.15}$$

In fact $\tilde{\kappa}$ must be zero. Otherwise we would have

$$K(u) = \frac{F(u)}{f(u)} = \frac{F(u)}{F'(u)} \geq \kappa, \qquad 0 < u < a$$

for some constant $\kappa > 0$. By integration

$$\frac{F(v)}{F(u)} \leq \exp\,(v-u)/\kappa, \qquad 0 < u < v < a.$$

Rewriting gives $|F(v)| \leq |F(u)|\,\exp\,(v-u)/\kappa$, whence letting $u \to 0$ yields $|F(v)| = 0$, $0 < v < a$, which is impossible. Hence (4.15) holds with $\tilde{\kappa} = 0$.

We now apply Lemma 3.2. To begin with, by (3.2) and (4.15), and using the notation $r = t(u)$,

$$\lim_{u\to 0} r^{n-1}\Omega(\rho(r))K(u) = \lambda \lim_{u\to 0} K(u) = 0. \tag{4.16}$$

Let $(r_i)_i$ be a sequence with $r_i \to \infty$ such that[4]

$$\lim_{r_i\to\infty} r_i^n[H(\rho_2(r_i)) + F(u_2(r_i))] = 0, \tag{4.17}$$

which exists in view of (3.3). Then putting

$$\varepsilon_i = u_2(r_i), \qquad i = 1, 2, \ldots,$$

[4]For a compact support solution (see the note after Lemma 3.2) we take $r_i \to b$.

we get with the help of (4.16) and (4.17) – note $r_i = t_2(\varepsilon_i) = R_{2\varepsilon_i}$ –

$$\begin{aligned}\liminf_{\varepsilon=\varepsilon_i\to 0}\{P(R_{1\varepsilon},\varepsilon,\rho_1(R_{1\varepsilon})) - D\,P(R_{2\varepsilon},\varepsilon,p_2(R_{2\varepsilon}))\} \\ = \liminf_{\varepsilon=\varepsilon_i\to 0} R^n_{1\varepsilon}[H(\rho_1(R_{1\varepsilon})) + F(u_1(R_{1\varepsilon}))] \geq 0.\end{aligned} \tag{4.18}$$

On the other hand, since the integrands on the right hand side of (4.10) are non–negative, the right hand side of (4.10) clearly approaches a a non-positive limit as $\varepsilon \to 0$. Consequently, in view of (4.18) the right hand side of (4.10) approaches zero as $\varepsilon \to 0$, and in turn the integrands on the right hand side must vanish on $(0,\alpha)$ and (α, a), respectively. Hence (see e.g. (4.6)–(4.8))

$$K'(u) \equiv \frac{n-m}{nm}.$$

for $0 < u < \alpha_2$, $u \neq a$. But this is impossible since $K(u) = F(u)/f(u)$ and

$$\lim_{u\to a^-} F(u)/f(u) = \infty.$$

We thus conclude that the graphs of the solutions $u_1(r)$ and $u_2(r)$ do not intersect at *any* point in the region $r \geq 0,\ u > 0$, that is

$$t_2(u) > t_1(u), \qquad 0 < u < \alpha.$$

This is itself impossible, however, as noted in both [FLS] and [ET]. Indeed by Lemma 3.3

$$t_2'(u) - t_1'(u) < 0, \qquad 0 < u < \alpha.$$

But this is an absurdity because, as noted earlier, $t_1'(u) \to -\infty$ as $u \to \alpha^-$ while $t_2'(\alpha)$ is finite.

This completes the proof of Theorem 1.

5. The canonical case $f(u) = -u^p + u^q$, $0 < p < q$.

Here (a), (b), (c) are satisfied with $a = 1$. We first consider the important case when $1 < m < n$.

As in Section 4 of Erbe and Tang, one computes that

$$K(u) = \frac{F(u)}{f(u)} = \left(-\frac{u^{p+1}}{p+1} + \frac{u^{q+1}}{q+1}\right) \Big/ (-u^p + u^q)$$

(so $K \to 0$ as $u \to 0$ without further argument!) and

$$K'(u) - \frac{n-m}{nm} = \frac{1}{\sigma+1} \cdot \frac{\xi v^2 - \zeta v + \nu}{(v-1)^2}$$

for $v \neq 1$, where

$$\sigma = \frac{(m-1)n+m}{n-m}, \qquad v = u^{q-p}$$

and

$$\xi = \frac{\sigma - q}{q+1}, \qquad \nu = \frac{\sigma - p}{p+1}, \qquad \zeta = \xi(q-p+1) + \nu(p-q+1).$$

Thus to satisfy (2.5) it is necessary to determine the range of exponents p, q such that

$$D(x) \equiv \xi x^2 - \zeta x + \nu \geq 0 \qquad \text{for } x \geq 0.$$

Naturally one must require $0 < p < q \leq \sigma$, a condition which we assume in the sequel without further mention. Then $D(x) \geq 0$ for $x \geq 0$ if and only if either

$$\zeta \leq 0 \qquad \text{or} \qquad \zeta^2 \leq 4\xi\nu.$$

Lemma 1. *$\zeta^2 \leq 4\nu\xi$ if and only if*

$$I(p,q) \equiv (p+1)(q-p+1)^2(\sigma-q) - (q+1)(p-q+1)^2(\sigma-p) \geq 0.$$

Proof. By direct calculation

$$\begin{aligned}
4\xi\nu - \zeta^2 &= 2\xi\nu(1+(q-p)^2) - \xi^2(q-p+1)^2 - \nu^2(p-q+1)^2 \\
&= -(\xi-\nu)^2\{(q-p)^2+1\} - 2(q-p)(\xi^2-\nu^2) \\
&= (\xi-\nu)\{-\xi((q-p)+1)^2 + \nu((p-q)+1)^2\} \\
&= \frac{(\sigma+1)(q-p)}{(p+1)^2(q+1)^2}\, I(p,q).
\end{aligned}$$

Lemma 2. *$D(x) \geq 0$ if $p+1 \leq q$.*

Proof. We may suppose $\zeta > 0$ without loss of generality. Then

$$(p+1)(1+q-p)(\sigma-q) + (q+1)(1+p-q)(\sigma-p) > 0$$

and, noting that $q-p-1 \geq 0$,

$$\begin{aligned}
I(p,q) &= (p+1)(1+q-p)^2(\sigma-q) + (q-p-1)\cdot(q+1)(1+p-q)(\sigma-p) \\
&\geq (p+1)(1+q-p)^2(\sigma-q) - (q-p-1)\cdot(p+1)(1+q-p)(\sigma-q) \\
&= 2(p+1)(\sigma-q)(1+q-p) \geq 0.
\end{aligned}$$

Lemma 3. *$I(p,q)$ for fixed p is a cubic in q with leading term $-(\sigma+1)q^3$. Moreover $I(p,p)=0$ and*

$$\begin{aligned}\frac{\partial I}{\partial q}(p,p) &= -4p^2+4(\sigma-1)p+3\sigma-1,\\ I(p,p+1) &= 4(p+1)(\sigma-p-1),\\ I(p,\sigma) &= -(\sigma+1)(\sigma-p)(\sigma-p-1)^2.\end{aligned}$$

Proof. Direct verification.

Lemma 4. *Let $\sigma>1$. Then $I(p,q)\geq 0$ when $p\leq\sigma-1$, $p\leq q\leq p+1$.*

Proof. From the first displayed line of Lemma 3 it is easy to see that $(\partial I/\partial q)(p,p)>0$ for $0<p<p_0(\sigma)$, where

$$p_0(\sigma)=\frac{1}{2}\left\{\sigma-1+\sqrt{(\sigma-1)^2+3\sigma-1}\right\}$$

(thus, e.g., when $n=3$, $m=2$ we have $\sigma=5$ and $p_0=4.739$).

Obviously $\sigma-1<p_0<\sigma$. Hence, for $p\leq\sigma-1$,

$$\frac{\partial I}{\partial q}(p,p)>0, \qquad I(p,p)=0, \qquad I(p,p+1)\geq 0.$$

Therefore, using the fact that I is a cubic in q with leading term $-(\sigma+1)q^3$, it follows easily that

$$I(p,q)\geq 0, \qquad p\leq q\leq p+1.$$

Lemma 5. *Suppose $\sigma-1<p<p_0$, $\sigma>1$. Then there exists $q_0=q_0(p,\sigma)$ such that $p<q_0<\sigma$ and*

$$I(p,q)>0 \qquad \textit{for} \quad p\leq q\leq q_0.$$

Proof. Similar to the previous lemma, except that the relation $I(p,p+1)\geq 0$ must be replaced by $I(p,\sigma)<0$.

Lemma 6. *Suppose* $1/3 < \sigma \leq 1$. *Then* $0 < p_0 < \sigma$. *Moreover, there exists* $q_1 = q_1(p, \sigma)$ *such that* $p < q_1 < \sigma$ *when* $0 < p < p_0$ *and*

$$I(p,q) \geq 0 \qquad \textit{for} \quad 0 < p < p_0, \quad p \leq q \leq q_1.$$

Proof. This is the same as for Lemma 5, once one notes that $p_0 > 0$ (rather than $p_0 > \sigma - 1$) because $\sigma > 1/3$.

Let $\sigma > 1$. Combining Lemmas 1, 2, 4, 5, we see that $D(x) \geq 0$ for $x \geq 0$ when $p \leq \sigma - 1$, and also when $\sigma - 1 < p < p_0(\sigma)$, $p < q \leq q_0(p, \sigma)$. This completes the proof of Theorem 2 for the case $n > m$. Theorem 2′ follows in the same way, using Lemma 6 rather than Lemmas 4, 5.

The case $n \leq m$ is easist considered by letting $\sigma \to \infty$ in (2.7), leading to the results noted immediately after the statement of Theorem 2.

6. Remarks and generalizations

1. Condition (a), that f be continuous in $[0, \infty)$ and $f(0) = 0$, can be weakened, provided some care is exercised concerning the meaning of compact support solutions, i.e. solutions such that $u(r) > 0$ for $r < b$ and $u(r) = u'(r) = 0$ for $r \geq b$. In particular, when $f(0)$ is *not* zero the function $u(r)$ in these cases can no longer be a solution of (2.1) when $r > b$. To avoid this difficulty we *define* a radial compact support ground state as a function $u(r)$, such that $u(r)$ is positive and satisfies (2.1) for $r < b$, has the property

$$u(r),\ u'(r) \to 0 \qquad \text{as} \quad r \uparrow b,$$

and finally $u(r),\ u'(r) \equiv 0$ for $r \geq b$.

Such compact support ground states exist even when $f(0)$ is not defined (see [KK] when $f(0)$ exists but is negative rather than zero); moreover one finds exactly as in [FLS], Lemma 1.2-6 (i), that they retain the property $u'(r) < 0$ for $0 < r < b$. The preceding uniqueness proof then continues to hold if we replace condition (a) by the weaker hypothesis

(a′) f is locally integrable on $[0, \infty)$.

In particular, the integral $F(u) = \int_0^u f(\tau)d\tau$ then exists, and $F(u) \to 0$ as $u \to 0$ (note that this applies even if $f(u) \to -\infty$ as $u \to 0$).

For the model nonlinearity

$$f(u) = -u^p + u^q \tag{1.2}$$

we can thus allow $-1 < p < q$. (The case $p \le -1$ is ruled out, since then f is not locally integrable on $[0, \infty)$). The uniqueness condition (2.5) for the nonlinearity (1.2) has already been explored in Section 4 when $p > 0$. For the range $-1 < p \le 0$, $\sigma \ge 1$ a similar argument applies, and in particular (2.5) holds when $0 \le q < \sigma$. To demonstrate this, it is enough to verify that

$$I(p, 0) > 0 \qquad \text{for} \quad -1 < p \le 0,\ \sigma \ge 1/3,$$
$$I(p, p+1) > 0 \qquad \text{for} \quad -1 < p \le 0,\ \sigma \ge 1,$$

and to use the cubic property of $I(p, q)$ together with $I(p, p) = 0$..

We summarize the above results in the following two theorems.

Theorem 3. *Let conditions* (a′), (b), (c) *hold. Then equation* (1.5) *admits at most one radial ground state when* (2.5) *is satisfied.*

Theorem 4. *Suppose* $-1 < p \le 0$, $0 \le q < \sigma$, $\sigma \ge 1$. *Then there cannot be more than one radial ground state of equation* (1.5) *for the nonlinearity* (1.2).

Observe from the Corollary in Section 1..3 of [FLS] that any ground state in the case of Theorem 4 in fact has compact support (since $p + 1 < m$).

Finally, there is a corresponding result to Theorem 2′, which we state without proof.

Theorem 4′. *Let* $1/3 \le \sigma < 1$. *Then* (1.1) *admits at most one ground state for the nonlinearity* (1.2) *when*

(i) $-1 < p \le \sigma - 1, \qquad 0 \le q < \sigma$

(ii) $\sigma - 1 < p \le 0, \qquad 0 \le q \le q_1$.

2. Condition (b) can also be weakened to the requirement that $f(u)$ be locally Lipschitz continuous on $(0, \infty)$. In this case condition (2.5) must be interpreted as holding almost everywhere, or alternatively (as one easily sees) that the function

$$K(u) - \frac{n-m}{nm} u$$

is non-decreasing on $(0, a)$ and on (a, ∞). The proofs remain identical, since by Proposition 1 the function P is still absolutely continuous when $u \neq a$ (rather than being continuously differentiable), which leaves the integral formulas (4.6) and (4.10), and so the proof itself, unchanged.

3. The equation

$$\Delta_m u + r^l f(u) = 0 \qquad \text{in } \mathbb{R}^n, \tag{6.1}$$

where

$$m > 1, \qquad l + m \min\left\{1, \ \frac{n-1}{m-1}\right\} > 0, \tag{6.2}$$

can be treated by the same approach.

Because of the singular term r^l in (6.1), it is however first necessary to give an appropriate definition of a "regular" solution of (6.1). That is, we consider non-negative radial distribution solutions of class $C^1(\mathbb{R}^n \setminus \{0\})$ which are *bounded* near $x = 0$, and with a further technical assumption if $n < m$, namely

$$Du(x) = o\left(r^{-(n-1)/(m-1)}\right) \qquad \text{as } x \to 0. \tag{6.3}$$

We shall also need the following

Lemma. *Let $u = u(r)$ be a regular radial solution of* (6.1)*, with* $\liminf_{r\to 0} u(r) = \alpha > 0$. *Then*

$$u'(r) = O\left(r^{(l+1)/(m-1)}\right) \quad \textit{and} \quad u(r) - \alpha = O\left(r^{(l+m)/(m-1)}\right) \tag{6.4}$$

as $r \to 0$, where $l + m > 0$ by (6.2)*.*

Proof. We proceed as in Lemma 1.1.1 of [FLS]. Thus, letting $w(r) = |u'(r)|^{m-2}u'(r)$, $r > 0$, it follows from (6.1) that

$$\left[r^{n-1}w(r)\right]' = -r^{l+n-1}f(u(r)), \qquad r > 0.$$

By integration over $[s, r]$, $0 < s < r$, there results

$$r^{n-1}w(r) - s^{n-1}w(s) = -\int_s^r t^{l+n-1} f(u(t))dt. \tag{6.5}$$

Clearly $l+n>0$ by (6.2), so that $t^{l+n-1}f(u(\cdot))\in L^1[0,\delta]$ for some $\delta>0$. Then (6.5) shows that $s^{n-1}w(s)$ tends to a finite limit γ as $s\to 0$.

We assert that $\gamma=0$. This is obvious from (6.3) if $n<m$. Suppose $n\geq m$ and, for contradiction, $\gamma\neq 0$. Then clearly $u'(r)=\operatorname{sign}\gamma|\gamma r|^{-(n-1)/(m-1)}[1+o(1)]$ as $r\to 0$. But $r^{-(n-1)/(m-1)}\notin L^1[0,1]$, so in turn u cannot be bounded near $r=0$. This is impossible, and the claim is shown.

Again letting $s\to 0$ in (6.5) we get

$$r^{n-1}w(r)=-\int_0^r t^{l+n-1}f(u(t))dt,$$

which implies

$$u'(r)=O\left(r^{(l+1)/(m-1)}\right)\qquad\text{as } r\to 0.$$

A further integration then gives $u(r)-\alpha=O(r^{(l+m)/(m-1)})$, completing the proof of (6.4).

Now, with the change of variable

$$y=\frac{m}{l+m}r^{(l+m)/m}$$

it is not hard to see that regular radial solutions of (6.1) satisfy the ordinary differential equation

$$(|u'|^{m-2}u')'+\frac{N-1}{y}|u'|^{m-2}u'+f(u)=0,\qquad y>0,\tag{6.6}$$

where primes now denote differentiation with respect to y, and

$$N=m\frac{l+n}{l+m}$$

(the restriction (6.2) on l implies that $l+m>0$ and $N>1$). Moreover, by (6.4) one finds

$$u'(y)=r^{-l/m}O\left(r^{(l+1)/(m-1)}\right)=O\left(y^{1/(m-1)}\right),\qquad u(y)-\alpha=O\left(y^{m/(m-1)}\right)$$

as $y\to 0$. That is, if $u=u(r)$ is a regular radial ground state of (6.1) then $u=u(y)$ is radial distribution solution of (6.6) of class $C^1(\mathbb{R}^n)$, satisfying the ground state conditions

$$u(y)\geq 0,\qquad u\not\equiv 0,\qquad u(y)\;\to\;0\quad\text{as } y\to 0.$$

Since

$$\frac{N-m}{Nm} = \frac{n}{l+n} \cdot \frac{n-m}{nm}$$

the uniqueness criterion (2.5) takes the form

$$\frac{d}{du}\frac{F(u)}{f(u)} \geq \frac{n}{l+n} \cdot \frac{n-m}{nm}.$$

That is, under this condition, *regular radial ground states of* (6.1) *are unique* – translations of the origin not being allowed since O is already fixed in (6.1).

Note that when $n > m$ the natural Sobolev exponent for (6.1) and (6.6) is

$$\sigma = \frac{(m-1)N+m}{N-m} = \frac{(m-1)n+(l+1)m}{n-m},$$

cf. (1.4).

7. The exterior Neumann problem.

Consider non-negative radial solutions of (1.5) in the exterior of the ball B_0 of radius R_0, subject to the boundary conditions

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B_0, \qquad u(r) \to 0 \quad \text{as } r \to \infty.$$

The required equation is then (2.1), with the conditions

$$u'(R_0) = 0, \qquad u(r) \to 0 \quad \text{as } r \to \infty.$$

The uniqueness question for this problem can be treated essentially as before, and all the previous results *continue to hold.*

In particular, to prove the analogue of Theorem 1, we assume for contradiction that there are two different solutions $u_1 = u_1(r)$, $u_2 = u_2(r)$ for $R_0 \leq r < \infty$. We may suppose

$$u_1(R_0) = \alpha_1, \qquad u_2(R_0) = \alpha_2, \qquad \text{with } a < \alpha_1 < \alpha_2.$$

The preceding arguments then carry over essentially unchanged, except for the proof that solutions cannot intersect above the line $u = a$.

More specifically, for the present case the relations (4.4) must be replaced by

$$P(R_{1c}, u_c, \rho_1(R_{1c})) - R_0^n F(\alpha_1) = n \int_{R_0}^{R_{1c}} r^{n-1} \rho_1 \Omega(\rho_1) L(u_1, \rho_1) dr,$$
$$P(R_{2c}, u_c, \rho_2(R_{2c})) - R_0^n F(\alpha_2) = n \int_{R_0}^{R_{2c}} r^{n-1} \rho_2 \Omega(\rho_2) L(u_2, \rho_2) dr.$$

Multiplying the second of these equations by C, as previously, and subtracting from the first now yields the relation (4.6), *modified* however by the inclusion of the additional term

$$S = R_0^n [C F(\alpha_2) - F(\alpha_1)]$$

on the left hand side.

We show that $S > 0$. To see this, note first that $C = [t_1(u)/t_2(u)]^{n-1} > 1$, and that (since $\alpha_2 > \alpha_1 > a$)

$$F(\alpha_2) - F(\alpha_1) = \int_{\alpha_1}^{\alpha_2} f(\tau) d\tau > 0.$$

It is therefore enough to have $F(\alpha_2) > 0$. This however follows from [FLS], Lemma 1.2.1 – which for its derivation relies only on the fact that $\rho = 0$ at $r = 0$, or in the present case, that $\rho = 0$ when $r = R_0$.

Having shown that $S > 0$, the argument proving that there cannot be two intersections above the line $u = a$ now carries over word for word.

Continuing with the modified proof, one finds next that the relation (4.10) is changed by the addition of the term

$$S' = R_0^n [D F(\alpha_2) - F(\alpha_1)]$$

on the left hand side.

We show that $S' > 0$, this now requiring only that $D > 1$, as in the previous argument. To obtain this, first note from Lemma 3.2 that

$$r^{n-1} \Omega(\rho_1) = \lambda_1 - \int_r^{\infty} r^{n-1} f(u_1) dr,$$
$$r^{n-1} \Omega(\rho_2) = \lambda_2 - \int_r^{\infty} r^{n-1} f(u_2) dr,$$

where $\lambda_1,\ \lambda_2 \geq 0$ are the limits in (3.2), respectively for the solutions $u_1 = u_1(r)$ and $u_2 = u_2(r)$. Changing variables from r to $u(r)$ in these relations, and subtracting, gives

$$[t_1(u)]^{n-1}\Omega_1(u) - [t_2(u)]^{n-1}\Omega_2(u) = \lambda_1 - \lambda_2 - \int_0^u \left\{ \frac{[t_1(u)]^{n-1}}{\rho_1(u)} - \frac{[t_2(u)]^{n-1}}{\rho_2(u)} \right\} f(u)du$$

in an obvious notation.

By (4.9) the expression in braces in the integrand is positive when $0 < u < U$, while $f(u) < 0$ for $0 < u < a$. Hence for $0 < u \leq a$ we get

$$[t_1(u)]^{n-1}\Omega_1(u) - [t_2(u)]^{n-1}\Omega_2(u) > \lambda_1 - \lambda_2$$

(recall that $U > a$). Also, by [FLS], Lemma 3.6.2, one obtains $\lambda_2 \leq \lambda_1$ since $u_2(r) \leq u_1(r)$ for all sufficiently large r. Consequently

$$[t_1(a)]^{n-1}\Omega_1(a) - [t_2(a)]^{n-1}\Omega_2(a) > 0,$$

which is exactly the condition $D > 1$.

Hence $S' > 0$, and the argument showing that there can be no intersections above the line $u = a$ carries over essentially unchanged.

Since no further modifications are required, Theorem 1 thus holds for the exterior Neumann problem, as well as for the ground state problem. The remaining theorems of the paper, dealing with the validity of condition (2.5) for the canonical nonlinearity (1.2), are of course unchanged, whether one is concerned with the ground state problem or the exterior Neumann problem.

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